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COLOR GROUPS OF SYMMETRICAL OBJECTS

PRIMES: FORMULAS and CONWAY'S MACHINE

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COVER: Two-color design by Hajime Ōuchi, in *Japanese Optical and Geometrical Art*, Dover, 1973. See p. 2.

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New manuscripts should be sent to: Doris Schattschneider, Editor, Mathematics Magazine, Moravian College, Bethlehem, PA 18018. Manuscripts should be prepared in a style consistent with the format of *Mathematics Magazine*. They should be typewritten and double spaced on $8\frac{1}{2}$ by 11 paper. Authors should submit the original and one copy and keep one copy as protection against possible loss. Illustrations should be carefully prepared on separate sheets of paper in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Marjorie Senechal ("Coloring Symmetrical Objects Symmetrically") is professor of mathematics at Smith College and has been interested in patterns for many years. She became interested in their colorings by reading Caroline MacGillavry's *Symmetry Aspects of M. C. Escher's Periodic Drawings,* and subsequently has written several research articles on the group-theoretic basis of color symmetry.

ILLUSTRATIONS

Margaret Huggard sketched the *Prime observations* appearing on pp. 22, 25, and 37.

David Logothetti pictured the prime-producing machine on pp. 32–33, with *Conway* supplying pedal power and many *Guys* inspecting.

John Beidler produced the computer-drawn graph for solution 1134, p. 50.

The 2-color cover design, by **Hajime Ōuchi** has (allowing for imperfections) symmetry group D_8. Four rotations and four reflections in this group interchange black and white, while the rest leave colors unchanged. See p. 3ff.

All other illustrations were provided by the authors.

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Coloring Symmetrical Objects Symmetrically

Classification of colored objects gives vivid interpretation of concepts in group theory.

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> *The pattern-maker often enjoys creating classes of motifs which are alike in one respect and different in another. Colouring is his most elementary device in achieving this end.*
> —E. H. Gombrich,
> **The Sense of Order**

> *There is a tremendous wealth of analogy in nature, and ... color symmetry gives us the means of expressing it.*
> —A. L. Loeb,
> **Color Symmetry and Its Significance for Science**

Color symmetry—the symmetrical distribution of colors in regular patterns—is as old as ornamental art itself. Beautiful examples from many cultures can be found in the colored plates of Owen Jones' classic *Grammar of Ornament* [**12**] and, in our time, in the tessellations of M. C. Escher. Also striking are the patterns of "identity and difference" [**2**] that are found in nature, for example in the arrangements in crystals of different atoms, or of magnetic spins [**8**]. Colors are often used in structure models to represent such nongeometric characteristics. In this article, we give an introduction to the mathematical theory of color symmetry that has been developed in recent years. This theory complements and extends the usual characterization of the symmetry of an object by describing the ways of coloring it that are consistent with its symmetry.

In addition to being of interest in its own right and for its applications, color symmetry provides a simple way of illustrating concepts in elementary group theory, such as subgroup, coset, normality, conjugacy, and so forth; we hope it will turn out to be a useful topic in introductory algebra courses. More advanced students will find that the theory of permutation groups provides a unified framework for examining various aspects of color symmetry; reformulating in more abstract terms the theory outlined here is an instructive exercise.

As an example of the problem of coloring an object symmetrically, suppose we wish to color each face of an octahedron with one of two colors, say, black and white. Intuitively we expect that in a symmetrical distribution of colors, four faces should be black and four white. There are many

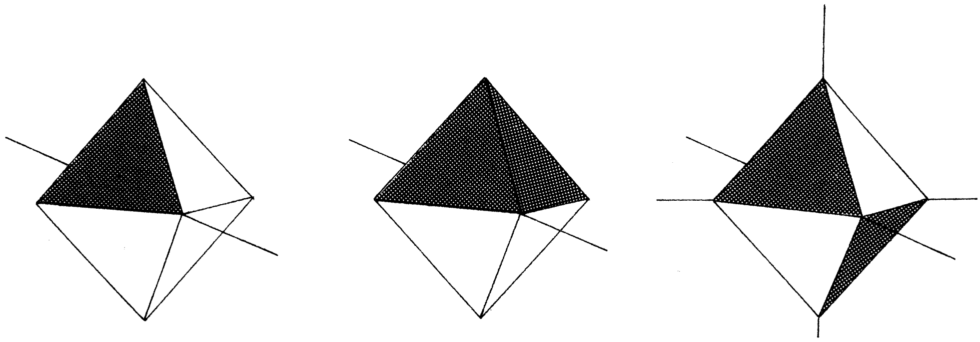

FIGURE 1. **Three of the many ways of coloring the faces of a regular octahedron with two colors. Only the coloring in (c), in which each face is adjacent only to faces of the other color, is consistent with the octahedron's symmetries.**

ways to do this, as FIGURE 1 suggests, displaying varying degrees of "balance." Of the three colorings in FIGURE 1, only FIGURE 1(c) is completely consistent with all of the octahedron's symmetries in the sense that each symmetry operation causes a well-defined permutation of the colors. For example, a 90° rotation of the octahedron about any axis shown in FIGURE 1(c) maps *every* black face onto a white one and vice versa; in addition, reflecting the octahedron onto itself through any of the three planes that cut it into two pyramids also causes a similar interchange of black and white faces.Thus these symmetry operations give rise to the permutation black $\leftrightarrow$ white. On the other hand, a 120° rotation about an axis through the centroids of any pair of opposite faces induces the identity permutation. However, in both FIGURES 1(a) and 1(b) (no matter how the hidden faces are colored) there are symmetries which map some of the white faces onto white faces while simultaneously interchanging some white and black faces; such symmetries do not determine true permutations of the colors.

As another example of color symmetry, consider the familiar black and red checkerboard, which we will assume is a finite, representative fragment of an infinite repeating pattern (FIGURE 2). Here the coloring is "obviously" symmetrical. What does this mean? The checkerboard has many different kinds of symmetries; for simplicity, let us consider only the translations. Writing $\vec{t}_1$ for a translation by one unit to the right and $\vec{t}_2$ for a translation upward by one unit, we observe that both translations cause a corresponding interchange of the colors. More generally, the integral linear combination $\alpha\vec{t}_1 + \beta\vec{t}_2$ gives rise to the permutation black $\leftrightarrow$ red if $\alpha + \beta$ is odd, and to the identity permutation if $\alpha + \beta$ is even (FIGURE 2). In other words, the correspondence ϕ between translations and the color permutations they induce satisfies the condition

$$\phi(\alpha\vec{t}_1 + \beta\vec{t}_2) = \phi(\vec{t}_1)^{\alpha}\phi(\vec{t}_2)^{\beta}.$$

This says that group composition is preserved under the correspondence between translations of the checkerboard and their color permutations. We can generalize this idea and say that the coloring of an object is **consistent** when the mapping ϕ from the group of symmetries of the object into the group of permutations of the colors is a homomorphism. *This is the link between the mathematical theory and the aesthetic harmony of colored patterns.*

As our final introductory example, we pose a problem for the reader to solve: can the edges of a cube be consistently colored with four different colors? (The answer is given in the next section.)

The history of the mathematical theory of color symmetry began in the late 1920s when the concept of two-color symmetry was introduced to describe the symmetries of repeating patterns in a two-sided plane in three-dimensional space (see for example, [**10**]). The new concept did not attract much attention at that time. Later (1951), interest in symmetrical colorings was stimulated

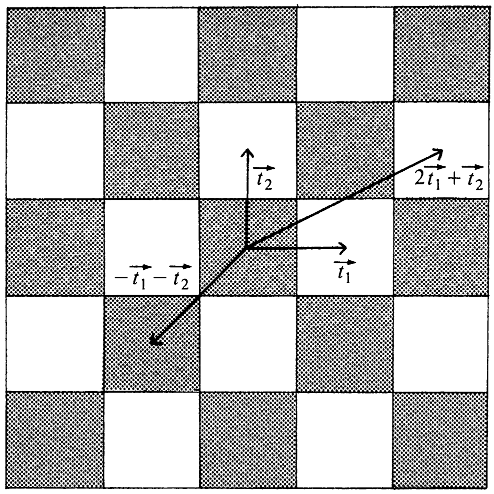

FIGURE 2. **A fragment of an infinite checkerboard. Its translation symmetries produce color permutations which satisfy the following rule: The color permutation associated with the sum of two translations is the product of the color permutations associated with the individual translations.**

by the Soviet crystallographer A. V. Shubnikov's monograph *Symmetry and Anti-Symmetry of Finite Figures* [**21**]. Shubnikov investigated polyhedra and other figures whose faces could be colored black and white by appending, when possible, an "anti-symmetry operation" (the permutation black ↔ white) to their symmetries. Because this use of color opened the door to refinements of the usual classification of spatial patterns by their symmetries, and thus to the solution of certain problems concerning atomic patterns in crystals, the idea was quickly extended by Shubnikov's colleagues to other groups of crystallographic interest, and to patterns with more than two colors [**21**]. Independently, the Dutch graphic artist M. C. Escher "invented" color symmetry to handle problems in background and foreground that he encountered in his work with plane tessellations. Crystallographers soon recognized the value of his colored tessellations as teaching aids. Caroline MacGillavry's *Symmetry Aspects of M. C. Escher's Periodic Drawings* [**15**] brought international attention both to Escher's work and to the developing theory of color symmetry [**13**]. In 1961, B. L. van der Waerden and J. J. Burckhardt showed that polycolor symmetry can be described in a simple way by the theory of permutation representations of groups [**22**]. Their work has been the basis for most of the recent work in the subject, including the present paper.

In the discussion that follows, we review the concept of symmetry, and then discuss its generalization to color symmetry and show how spherical tessellations (and therefore polyhedra) can be colored symmetrically. We then consider the problem of classifying consistent colorings of objects. We close with a brief discussion of the evolution of some of the concepts we have presented and suggest some possible directions for further investigation.

Symmetries and symmetry elements

The study of symmetry can be traced back at least as far as the discovery by the ancient Greeks of the five regular convex (Platonic) solids. These striking forms are characterized by the requirement that their faces are congruent regular polygons, the same number of which meet at each vertex. We readily perceive symmetry here in its intuitive sense: balance, harmony, equality of parts. However, we can also describe the symmetry of these polyhedra and other geometric objects in a more mathematical, formal, and abstract way.

The mathematical study of symmetry is concerned with the interplay of geometry and algebra through the action of symmetry groups on geometric objects. The **symmetry group** G of an object

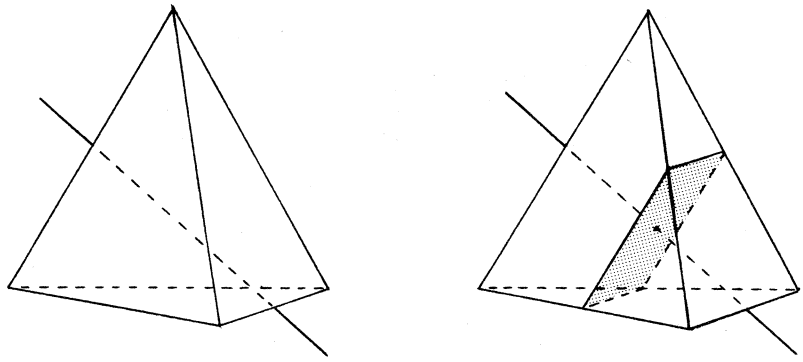

FIGURE 3. **(a) A rotation of 180° about an axis through the midpoints of opposite edges of a regular tetrahedron is a symmetry of the tetrahedron. (b) This rotation axis is also the axis of a 4-fold (90°) rotatory-reflection which is a symmetry of the tetrahedron. The reflection plane of this symmetry intersects the tetrahedron in a square through the midpoints of the other four edges.**

is the set of isometries (distance-preserving mappings) which bring the object into coincidence with itself—this set always forms a group under composition. The isometries in G are called the **symmetry operations**, or more briefly, the **symmetries** of the object. In this paper we will be primarily interested in the symmetries of polyhedra and tessellations on the surface of a sphere. For these cases G must be one of the finite groups of isometries of the sphere. These groups are listed and described in [**1**]. They include four types of isometries: rotation, reflection, central inversion, and rotatory-reflection. This last operation, which includes reflection and inversion as special cases, can be described as rotation followed (nonstop) by reflection in a plane perpendicular to the axis of rotation. To each symmetry g in G corresponds a set of points in Euclidean 3-space which are left fixed by g; this set is called the **symmetry element** corresponding to g. If g is a rotation, then its symmetry element is a line, the axis of rotation; if g is a reflection, then its symmetry element is the plane of reflection; if g is central inversion or a rotatory-reflection, then there is a single fixed point, the center of the object.

When attempting to identify the symmetries of an object, the location of rotation axes, reflection planes, and centers of inversion are usually easy to see. The symmetry of rotatory-reflection can be more difficult to spot. The following device is very helpful. We define an "axis of rotatory reflection" to be the axis of the rotation part of the rotatory-reflection (although, strictly speaking, such axes are not symmetry elements). These rotatory-reflection axes occur among the rotation axes since a rotatory-reflection composed with itself is a rotation (or the identity); a rotatory-reflection is a "square root" of a rotation. (Central inversion, a rotatory-reflection whose angle of rotation is 180°, is thus a square root of the identity!) Using this technique, one can see that the regular tetrahedron has an axis of 4-fold rotatory-reflection through the midpoints of each pair of opposite edges (FIGURE 3); the corresponding symmetry operation maps the four faces of the tetrahedron onto one another cyclically. (Can you locate the 6-fold axes of rotatory-reflection of the cube?)

Now suppose that G is the symmetry group of an object which contains a finite set X that G maps onto itself. (Typically, X might be the set of faces, or edges, or vertices of a polyhedron.) For each x in X, the set $\{gx | g \in G\}$ is called the ***G*-orbit** of x, where gx denotes the image of x under the action of g. The group G **acts transitively** on X if the G-orbit of any (alternatively: every) x in X is the set X itself, or equivalently, if for each ordered pair (x_i, x_j) in X there is at least one g in G such that $gx_i = x_j$. For example, this concept can be used to single out the regular polyhedra from all the others by the fact that their symmetry groups act transitively on their faces, edges, and vertices. From now on, *we will assume that G acts transitively on X*. For an element x in X, the

stabilizer of x is the subgroup $G_0(x)$ consisting of the isometries in G which leave x fixed: $G_0(x)=\{g \in G | gx = x\}$. It is a standard exercise to show that if g in G maps x_i to x_j, then g' in G also maps x_i to x_j if and only if g' belongs to the *left* coset $gG_0(x_i)$. Further, $gG_0(x_i)g^{-1}$ is the stabilizer $G_0(x_j)$ of x_j. Since the stabilizers of the elements of X are all conjugate, we can speak loosely of *the* stabilizer subgroup G_0 of X.

It is often useful to label the elements of X with elements of G, as follows. We arbitrarily choose one x in X to be the initial element x_e (e denotes the identity element of G), and then label the other elements of X with the symmetries which map x_e onto them, that is, according to the formula $x_g = gx_e$. Of course the elements of X do not always receive unique labels from this process. In the first place, the labeling depends upon the choice of x_e. Also, if the stabilizer subgroup G_0 is of order m, then since every element of the left coset $gG_0(x_e)$ maps x_e onto x_g, x_g receives $|G_0|$ labels. Thus $|X|=|G|/|G_0|$.

For example, consider the spherical tessellation shown in FIGURE 4 (spherical tessellations are discussed in more detail in the next section). Its symmetry group G, of order 12, is the group of rotations of the regular tetrahedron. Let X be the set of tiles of the tessellation, that is, the set of congruent regions into which the surface of the sphere is divided. The group G is transitive on X, and $G_0=\{e\}$; thus each tile is identified with a single element of G and receives a single label. However, if instead our object was a cube, for which $|G|=48$, and X was its set of faces (or edges or vertices), then each element of X would receive 8 (or 4 or 6) labels (FIGURE 5). In general, the interiors of the elements of X intersect the symmetry elements of the object if and only if the stabilizers of the elements are nontrivial. Thus in FIGURE 4 the rotation axes of the tessellation pass through the vertices of the tiles, not through their interiors; this agrees with the fact that, for those tiles, $G_0=\{e\}$. On the other hand, the interiors of each face of the cube (FIGURE 5(a)) intersect a 4-fold rotation axis and also four reflection planes.

With a little practice, the labeling procedure becomes fairly easy. Ping-pong balls make excellent experimental models, since spherical tessellations (or "blown-up" polyhedra) can easily be drawn on them, and the labels marked. (Use a pencil with an eraser!)

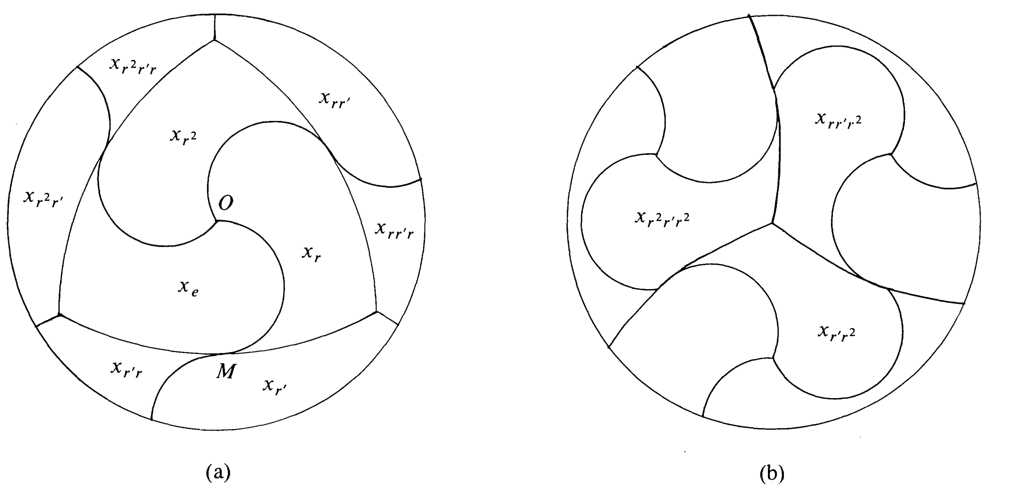

FIGURE 4. **A tessellation labeled with the elements of its symmetry group. If we inscribe a regular tetrahedron in a sphere, project it centrally onto the surface of the sphere, then join each face center (such as O) to the midpoints of the three edges of that face by semi-circles, we obtain the tessellation shown. Its symmetry group G coincides with the group of rotations of the regular tetrahedron. Since the rotation axes pass through the vertices of the tessellation, only the identity fixes any tile; thus $G_0=\{e\}$. Here we see a labeling of the tessellation by the elements of G; r is the 3-fold rotation whose axis joins O to the center of the sphere and r' is the 2-fold rotation whose axis joins M to the sphere's center. The front of the sphere is shown in (a). The tile labeled x_e is chosen arbitrarily; the others are labeled according to the formula $x_g = gx_e$. The reverse side of the sphere is shown in (b), as seen from the front, as if the sphere were transparent.**

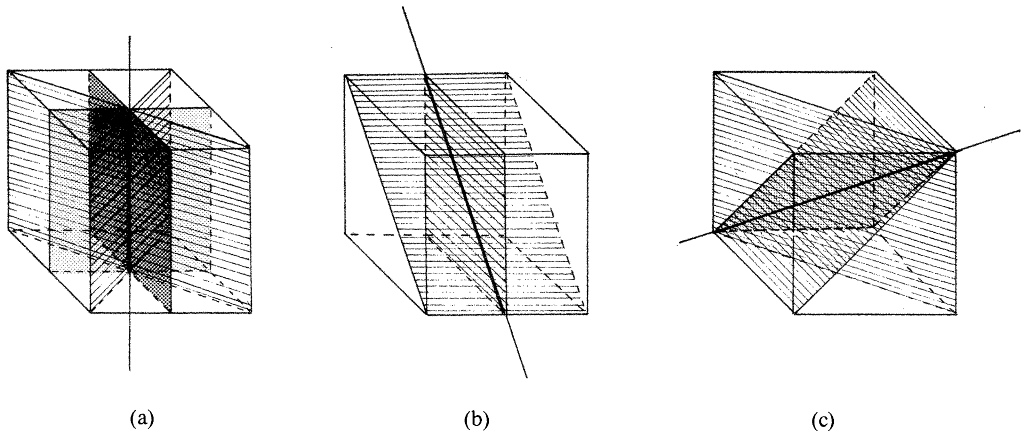

FIGURE 5. **The symmetry group of the cube acts transitively on its faces, edges, and vertices. (a) The stabilizer subgroup of a cube face is isomorphic to the dihedral group D_4: it contains four reflections together with four rotations about the line in which the reflection planes intersect. (b) The stabilizer of a cube edge is isomorphic to D_2, of order 4. (c) The stabilizer of a cube vertex is isomorphic to D_3.**

Perfect color symmetry

The mathematical formulation of symmetry we have just discussed can be extended in a natural way to incorporate the additional information of coloring. In the first section we described intuitively some of the properties of **color symmetry**; we now make these notions precise. As before, suppose we have an object containing a finite set X on which the symmetry group G of the object acts transitively. Let us assume that X is colored in such a way that each x in X receives one of k colors; we will denote the colored set by $\bar{X}$. The set $\bar{X}$ is said to be **perfectly colored** if every symmetry in G induces a unique permutation of the colors. In fact, this requirement ensures that $\bar{X}$ is partitioned into k congruent, single-color sets, since G is transitive on X and each g in G must map all the x's of one color to x's of another (single) color. It also guarantees that the mapping ϕ which associates with each g in G the permutation of the k colors induced by the action of g on X is a homomorphism from G into S_k, the group of permutations of k letters (colors). (It is not hard to see that the color permutation associated with a composite symmetry must be the composite of the associated individual color permutations.) Thus the coloring of a perfectly colored set is consistent with the symmetry group G, as defined earlier.

Some perfect 2-colorings were described in the first section. In FIGURE 6 we see four perfect 4-colorings of the spherical tessellation of FIGURE 4, and in FIGURE 8 part of a perfect 3-coloring of this same tessellation. Comparing FIGURE 6 with FIGURE 4, we see that although the colorings in FIGURES 6(a), (b), (c), (d) are different, in each case the 3-fold rotation r is associated by ϕ with the color permutation **(1)(243)**, and the 2-fold rotation r' with the permutation **(12)(34)**. Since r and r' generate G, the permutations associated with the remaining elements of G can be easily computed. It is important to note that the color **1** is fixed by all the permutations associated with the cyclic subgroup H generated by r, and only by those permutations. Not immediately evident from the figures, but easily verifiable with the help of a three-dimensional model, is the fact that the ϕ-image of the conjugate subgroup $r'Hr'^{-1}$ fixes color **2**, and the remaining single colors **3**, **4** are each fixed by the ϕ-image of a subgroup conjugate to H.

Comparing FIGURE 6(a) with FIGURE 4, we see that color **1** appears on the tiles labeled e, r, and r^2, which are the tiles labeled by elements of H. These tiles form an H-orbit. Color **2** appears on the tiles labeled r', $r'r$, and $r'r^2$ (the tile $r'r^2$ lies on the reverse side of the sphere). That is, color **2** appears on the tiles labeled with the elements of the *left* coset $r'H$. Similarly, colors **3** and **4** appear on the tiles labeled with the elements of the other two left cosets of H.

In general, in any perfect coloring of X the set of permutations which fix a given color, say **1**, is the ϕ-image of a subgroup H of G. For if $\phi(g)\mathbf{1}=\mathbf{1}$ and $\phi(g')\mathbf{1}=\mathbf{1}$, then

$$\phi(gg')\mathbf{1}=\phi(g)\phi(g')\mathbf{1}=\mathbf{1}.$$

(The subgroup H can be regarded as the stabilizer of the color **1**; it plays the same role with respect to the set of x's with that color as $G_0(x)$ plays with respect to x.) Again, as in the example above, if $\phi(g)$ carries **1** to **2**, so does $\phi(gH)$; conversely, if $\phi(g')\mathbf{1}=\mathbf{2}$, then g' belongs to gH, since $g^{-1}g'$ is in H. Thus each color is identified with a unique left coset of H, and the number of colors is equal to $[G: H]$, the index of H in G. It is also easy to show that if $\phi(g)$ carries **1** to **2**, then $\phi(gHg^{-1})$ fixes **2**; thus conjugate subgroups "act alike". (Compare these statements with the analogous statements about $G_0(x)$ on page 7.)

Even though H is defined as the subgroup of G whose ϕ-image fixes one color, it is quite possible for H to fix more than one color. The largest normal subgroup of G contained in H, $\cap_{g\in G}gHg^{-1}$, is the kernel of ϕ since its ϕ-image fixes every color. Thus $\phi(H)$ contains only the identity permutation (that is, $\phi(H)$ fixes all the colors) if and only if H itself is normal in G. If $\phi(H)$ fixes n colors so does $\phi(gHg^{-1})$; the sets of colors fixed by these two subgroups are disjoint if $gHg^{-1}\neq H$, and coincident if $gHg^{-1}=H$. As an example, consider a cube in which each face has a different color (FIGURE 7). Here r is a 4-fold rotation and r' is a 3-fold rotation. If H is defined as before, ($\phi(H)$ is the stabilizer of **1**), then in this case H is a dihedral group D_4 of order 8, the group of symmetries of a square. Since H fixes two faces of the cube (the top and bottom), $\phi(H)$ fixes two colors. The subgroups $r'Hr'^2$ and r'^2Hr' are conjugate to H; the corresponding permutations fix the pairs of colors $\{\mathbf{3},\mathbf{5}\}$ and $\{\mathbf{4},\mathbf{6}\}$ respectively. This observation gives us a very

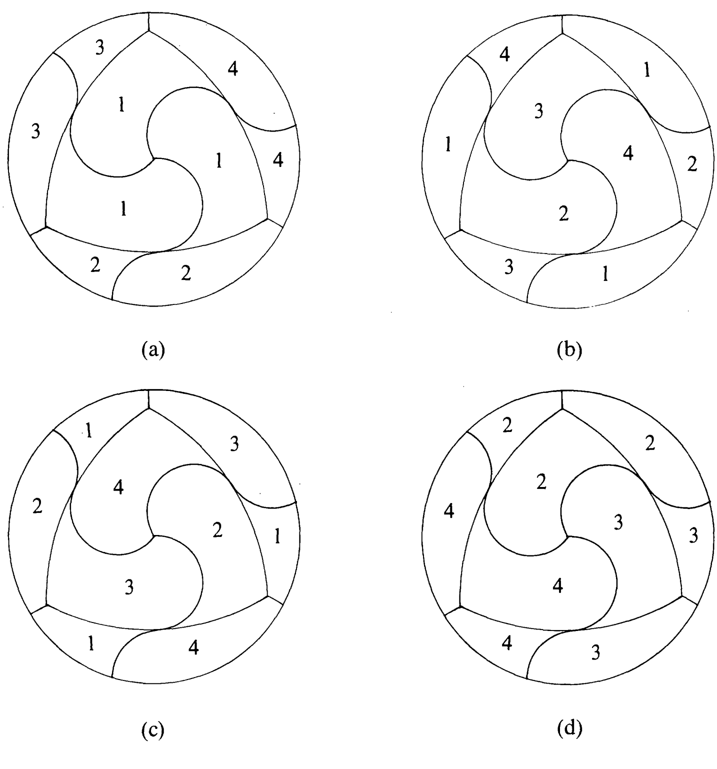

FIGURE 6. **Four perfect 4-colorings of the spherical tessellation in** FIGURE **4. In each case, $\phi(r)=(1)(243)$ and $\phi(r')=(12)(34)$. Therefore $\phi(rr')=(142)(3)$, and so forth. The subgroup H whose ϕ-image is the stabilizer of color 1 is the cyclic subgroup of order 3 generated by r. (How are the colors distributed on the reverse side of the sphere?)**

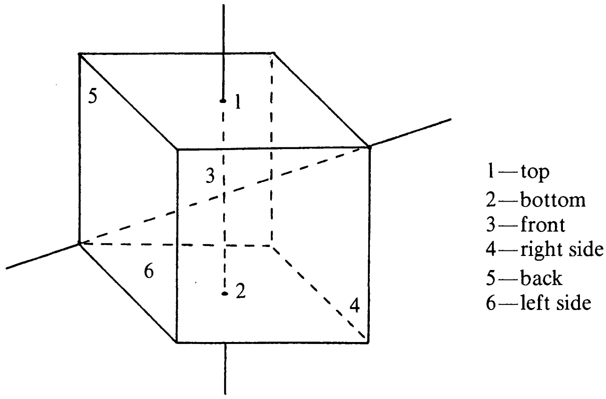

FIGURE 7. **A perfectly colored cube. The stabilizer H of one face is also the stabilizer of the opposite face, thus $\phi(H)$ fixes two colors. The subgroup H is isomorphic to D_4; it is generated by a 4-fold rotation r and the reflection in a plane containing the rotation axis (see** FIGURE **5(a)).**

simple way of finding the number of conjugates of the subgroup H: divide the total number of colors by the number of colors fixed by H.

The color permutations associated with the elements of G can be determined from a colored object, but they can also be found without reference to it if the subgroup H (the stabilizer of one color) is given. We identify the k left cosets $\{g_iH\}$, where $g_1 = e$, with the k colors. A symmetry operation g, performed on $\overline{X}$, effects the color permutation $\{g_iH \to gg_iH\}$. This defines a mapping ϕ from G into S_k, where

$$\phi(g) = \begin{pmatrix} H & g_2H & \dots & g_kH \\ gH & gg_2H & \dots & gg_kH \end{pmatrix};$$

clearly ϕ is a homomorphism, a permutation representation of G. The set of ordered pairs $\{(g, \phi(g)) | g \in G\}$ form a group under termwise composition, called a **k-color group** or simply **color group**, denoted by G_H. The notation G_H is intended to emphasize the fact that the color groups associated with G are determined by its subgroups.

We now ask: *which subgroups H of G correspond to perfect colorings of X?* Evidently in some cases certain subgroups do not. For example, since the regular tetrahedron has four faces, at most four colors can be used to color it, if each face is to receive a single color. Thus a subgroup of index > 4 will not correspond to a perfect coloring (nor, it is easily seen, will a subgroup of index 3). On the other hand, *any* subgroup of the group G of tetrahedral rotations defines a perfect coloring of the tessellation of FIGURE 4. To see this, we label the tiles with elements of G as shown in FIGURE 4. Let H be any subgroup of G, and color the tiles labeled by elements of H with color **1**. Then for some g in G, g not in H, color the tiles labeled by elements of gH with color **2**. Continue this process until all the tiles are colored. It is easy to verify that this is a perfect coloring. An example with $H = \{e, r', r^2r'r, rr'r^2\}$ is shown in FIGURE 8.

In general, if $G_0 = \{e\}$, then any subgroup H of G defines a perfect coloring in the above manner. But if $G_0 \neq \{e\}$, this may not work. We need to be sure that each x will receive a single color. If we apply the labeling procedure to a set X for which G_0 is nontrivial, then as we noted earlier, each x will receive $|G_0|$ labels. The subgroup H will define a perfect coloring if and only if each x which has at least one H-label has only H-labels, or equivalently, if $G_0(x) \subseteq H$. For the six-colored cube in FIGURE 7, for example, H is the stabilizer subgroup of one pair of opposite faces. Letting x be one of these faces, we have $G_0(x) = H$ and hence a perfect coloring is possible. However, if H is the subgroup of order 8 generated by the three conjugate reflections in planes perpendicular to the 4-fold cube axes, then each face receives only four H-labels, along with four others (FIGURE 9) and the faces of the cube cannot be perfectly colored by H.

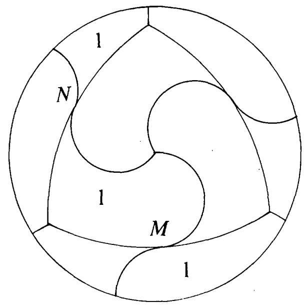

FIGURE 8. **The beginnings of a perfect 3-coloring of the tessellation in FIGURES 4 and 6. Let H be the subgroup of order 4 generated by the 2-fold rotations r' and $r^2r'r$ whose axes join points M and N, respectively, to the center of the sphere. The tiles labeled with elements of H have been assigned color 1. To continue this coloring, assign color 2 to one of the two left cosets of H, and color the tiles labeled with the elements of this coset accordingly. The remaining tiles all receive color 3. (Why?) Note that $\phi(H)$ keeps every color fixed; this shows that H is normal in G.**

As a final example, let us consider the question raised in the introduction: can the edges of a cube be perfectly 4-colored? Here X is the set of twelve edges. The stabilizer subgroup G_0 of an edge is the dihedral group D_2, consisting of a reflection in the perpendicular bisecting plane of an edge, a reflection in the diagonal plane containing that edge, and their product which is a half turn that permutes body diagonals. The symmetry group G of the cube has five subgroups of index 4. One of these, the group of rotations of the regular tetrahedron, contains no reflections. The other four are all conjugate; each consists of the set of symmetries which preserve a "body diagonal", that is, the stabilizer subgroup of a pair of opposite vertices of the cube together with the symmetries that map one of these vertices onto the other. From FIGURE 5 it is clear that G_0 is not contained in any of them. Thus there are no perfect 4-colorings of the edges of a cube.

The perfect colorings of polyhedra are of special interest because, when realized by suitable models, they provide tangible illustrations of important algebraic and geometric ideas. Fortunately, polyhedra (which can be, in any case, difficult to construct) can be considered to be special cases of tessellations of the sphere [4]. This generalization enables us to treat the "coloring problem" for polyhedra in a unified way.

The soccer ball is a familiar example of what is called a spherical tessellation. Formally, a **tessellation** of the sphere is defined to be a finite family of topological disks, called **tiles**, which cover the surface of the sphere without gaps and whose interiors are disjoint. The tiles need not be congruent: the spherical pentagons of the soccer ball are not congruent to the spherical hexagons.

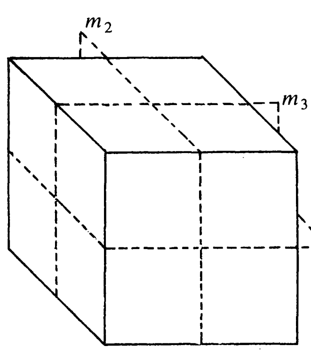

FIGURE 9. **If H is the subgroup of the symmetry group of the cube generated by reflections in the planes m_1, m_2, and m_3, then each cube face receives four H-labels and four others. If we try to color the cube by assigning color 1 to faces labeled with elements of H and the other colors to faces labeled with its cosets, each face will receive more than one color. Hence a perfect coloring is impossible.**

When discussing the symmetry group of a tessellation, X is usually assumed to be the set of tiles. However, the contexts in which spherical tessellations arise sometimes suggest that the edges or vertices of the tessellation, rather than the tiles, are the features of interest. (An edge is an arc which is the intersection of two tiles; a vertex is a point at which three or more tiles meet.) Examples include the coordination polyhedra studied in chemistry, which are defined by their vertices (namely, the positions of the atoms nearest to a given atom), and the many spherical tessellations described in [**23**], which are all edge constructions.

An important special class of spherical tessellations can be obtained by the central projection of polyhedra onto circumscribed or inscribed spheres. The tiles are then the projections of the polyhedral faces. But since all such tessellations have "straight" edges, many interesting tessellations cannot be obtained in this way. (FIGURE 4 shows a tessellation which is not the central projection of a polyhedron.) Ingenious examples can be obtained by projecting the tessellated polyhedra (but not the ring-shaped kaleidocycles) in *M. C. Escher Kaleidocycles* [**17**].

In order to classify spherical tessellations, we notice that the tiles define a spherical network of edges and vertices; the tessellations are classified first by their networks and then by their symmetry. (If we "straighten" the edges of the tessellation of FIGURE 4, we obtain a tessellation topologically equivalent to it, but one with a larger symmetry group G and a larger stabilizer subgroup G_0. Thus the two tessellations should be considered distinct.) A complete classification of spherical tessellations with transitivity properties was given recently by Grünbaum and Shephard in [**4**].

In order to determine the color groups of transitive spherical tessellations in which the set X is either the tiles, edges, or vertices of the tessellation, we consider the symmetry groups of the tessellations and look for "admissible subgroups H", that is, those subgroups which contain the stabilizer of a single tile, edge, or vertex of the tessellation. This does not have to be carried out for each tessellation separately. If we label the tiles or edges or vertices of the various tessellations with the symmetry operations of their symmetry groups, we see that often the same groupings of labels appear in the same positions relative to symmetry elements. For example, we note that the vertices of the octahedron receive the same labels as the faces of the cube, and vice versa. This is a great simplification, since "colorability" depends only on the constellations of labels; we are not concerned with the topology of the tessellations *per se*. The constellations of labels can be represented geometrically by a set of motifs of appropriate symmetry and position [**6**]. By finding the colorings of these "spherical patterns," the perfectly colored tessellations, and the colored polyhedra related to them, can be determined in a straightforward way [**20**].

Classifying perfect colorings

When should two perfect colorings of an object be considered the same, and when should they be considered different? The basic problem is to find a mathematical definition of equivalence which provides a classification appropriate for applications in mathematical or physical problems. Of course, a classification appropriate for one purpose may be inappropriate for another. We describe here the one that currently seems the most satisfactory to us.

This classification is a two-stage process. Given an object with a set X and symmetry group G which acts transitively on X, the first step is to enumerate the subgroups H of G giving rise to inequivalent color groups G_H. Then given a color group G_H, the second step is to describe the inequivalent colorings having G_H as their color group. To illustrate the process of classification by color group, consider a rectangular box divided into eight regions by its three reflection planes m_1, m_2, m_3 (FIGURE 10). Since the length, width and height of the box are all different, the colorings shown in FIGURE 10(a), (b), and (c) are different, yet they differ much more markedly from the coloring in FIGURE 10(d) than they do from one another. Each plane has associated to it a reflection which generates a subgroup of order 2 of the box's symmetry group G. Let H_1, H_2, and H_3 denote these subgroups. How are they related? Certainly H_1, H_2, and H_3 are isomorphic, but they are also isomorphic to the subgroups generated by the two-fold rotations of the box. Since these symmetries are not the same type, the color groups defined by these rotations (FIGURE

10(d)) ought to be considered distinct from the color groups defined by H_1, H_2, H_3. However, although H_1, H_2, and H_3 play equivalent roles in G, these three subgroups are not conjugate in G, so a requirement of conjugacy is too strong.

Let us examine more closely the statement that H_1, H_2, and H_3 "play equivalent roles in G". To do this, we ignore the box and consider only the symmetry elements of G. This configuration of three mutually perpendicular infinite planes has the symmetry group of the cube, which we will denote here by G^*. The group G^* contains G as a normal subgroup, and H_1, H_2, and H_3 are conjugate in G^*. This means that there are elements of G^* which map G onto itself by conjugation and at the same time map one of the H_i's onto another. It is in this sense that H_1, H_2, and H_3 are equivalent. In the case at hand, G is mapped onto itself under conjugation by the operation in G^* corresponding to reflection in the plane lying halfway between m_1 and m_2, and in this process H_1 is mapped onto H_2. Formally, then, we define two color groups G_H and $G_{H'}$ to be **equivalent** if there is an isometry α of E^3 which maps G onto itself and H onto H'. We will also say that H and H' are equivalent. (For infinite discrete groups G, α is usually taken to be an affine transformation.) Since such an α maps the cosets of H onto the cosets of H', the color permutations associated with elements of G are interchangeable. Two perfect colorings of an object are **equivalent by color group** if such an α can be found.

Color groups alone do not fully characterize perfect colorings, and this brings us to the second stage of the classification process. Let us consider the situation in FIGURE 6 first. There we saw four very different perfect colorings of the same spherical tessellation, all having the same color group. We have already seen that in FIGURE 6(a) color **1** appears on the tiles labeled by the elements of the cyclic group H of order 3 generated by r, and that these tiles form an H-orbit.

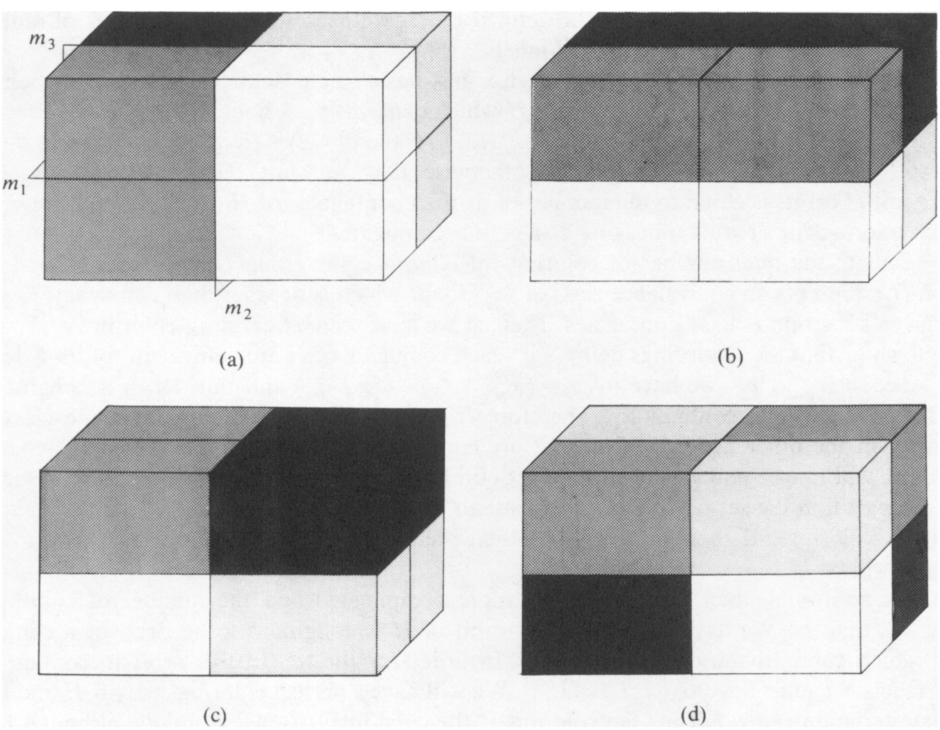

FIGURE 10. **Our definition of equivalence should be "natural." Thus the color groups associated with colorings (a)–(c) should be equivalent to each other but not equivalent to the color group of (d). This intuitive notion corresponds to the fact that in the first three, the single-color stabilizer subgroups H can be mapped onto one another by an automorphism of G, but not onto that of the fourth coloring.**

Now notice that in FIGURE 6(b) the color **1** appears on the tiles labeled r', rr', and r^2r', that is, on tiles labeled with elements of the *right* coset Hr'. These tiles constitute a different H-orbit, as do the tiles of color **1** in FIGURES 6(c) and 6(d). The differences between these colorings are thus due to a different choice of an H-orbit to form a single-color subset of the tiles.

In general, the H-orbit and thus a coloring depends on the choice of initial element x_e. This dependence can be very striking. In FIGURE 6(a) the rotation corresponding to the 3-fold axis passing through the vertex O of tile x_e maps this tile onto an adjacent tile of the same color; this is not the case in FIGURE 6(b). Similarly, if in some tessellation the tile x_e has an edge on a reflection plane which is the symmetry element of a reflection in H, then it will be adjacent to a like-colored tile under reflection in that plane; otherwise it will not. In physical applications, these differences can be very important.

We will say that two perfect colorings $\bar{X}$ and $\bar{X}'$, both realizations of the same color group G_H, are **equivalent as color configurations** if and only if $\bar{X}$ and $\bar{X}'$ can be obtained from one another by a one-one relabeling of colors. In other words, each single-color set of elements in $\bar{X}$ is also a single-color set in $\bar{X}'$.

Suppose that $\bar{X}$ and $\bar{X}'$ are equivalent color configurations corresponding to the color group G_H. We may assume that color **1** has been assigned to the H-orbits in each coloring. Choose x_e to be an x which has color **1** in $\bar{X}$ and label the elements of X in $\bar{X}$ accordingly. Next, label the elements of $\bar{X}'$ exactly as they are labeled in $\bar{X}$. Then since the set S of x's in X which have color **1** in $\bar{X}'$ is an orbit for H, S is labeled with the elements of a right coset Hg. But in $\bar{X}$, S is also a single-color set, since $\bar{X}$ and $\bar{X}'$ are equivalent. Thus in $\bar{X}$, S is labeled with the elements of a left coset $g'H$. Therefore $g'H = Hg$. Geometrically this equation says that g', a symmetry of X, maps H onto Hg. That is, the orbit labeled H, and S, the orbit labeled Hg (the sets of elements of X with color **1** in $\bar{X}$ and in $\bar{X}'$) are congruent under the symmetry g'. Therefore, in order to enumerate the number of colorings of X defined by H, we need to find the number of mutually incongruent H-orbits which carry only H labels.

First let us assume that $G_0 = e$, so that each x in X has a single label. Suppose that a labeling of X by G is given and let $\{x_h\}$ be the orbit of H which contains x_e. Choose x_g not in this orbit and let $\{x_{hg}\}$ be the orbit of H containing it. If $g^{-1}Hg = H$, then $\{x_{hg}\} = \{x_{gh}\} = \{gx_h\}$ and so the two H-orbits $\{x_h\}$ and $\{x_{hg}\}$ are congruent; otherwise they are not. Therefore the number of incongruent H orbits is equal to the number of distinct conjugates of H (including H) in G; there are four colorings in FIGURE 6 because H has four conjugates.

To calculate the *total* number of colorings of X with color group G_H, we recall that in this symbol, H represents an equivalence class of subgroups which is broader than conjugacy. Suppose that a given subgroup H has s conjugates. Then, as we have seen, H defines s colorings of X, and it is easy to show that the s colorings defined by each conjugate of H are equivalent to those defined by H: since $hg = gg^{-1}hg$, we have $\{x_{hg}\} = \{x_{g(g^{-1}hg)}\} = \{gx_{g^{-1}hg}\}$ and thus $\{x_{hg}\}$ is congruent to the orbit $g^{-1}Hg$ which contains x_e. Therefore H and its conjugates define a single class of s colorings. On the other hand, if H and H' are equivalent but not conjugate, then the colorings they define will not be equivalent under our definition (see FIGURE 10(a), (b), (c)). This suggests that if we partition the set of subgroups equivalent to H into conjugacy classes, each of them will represent s colorings. If there are r classes, then the total number of colorings of X with color group G_H will be rs.

If G_0 is nontrivial, then the situation is more complicated and the number of colorings is usually less than rs. We have seen that each orbit of H is congruent to an orbit of a conjugate gHg^{-1} which contains the element x_e of X. In order for the x's in this orbit to contain only gHg^{-1} labels, we must have $G_0(x_e) \subseteq gHg^{-1}$. We will have s perfect colorings only if H and all its conjugates contain $G_0(x_e)$. Thus the coloring of the cube in FIGURE 7 is unique although H has three conjugates. If H' is equivalent but not conjugate to H, then we obtain another set of s perfect colorings if and only if $G_0(x_e)$ is contained in H' and all its conjugates, and so on. All in all, the number of colorings admitted by G_H is equal to the number of distinct subgroups equivalent to H which contain $G_0(x_e)$.

Some comments on the past and the future

It is interesting that although congruence *between* objects has been a fundamental concept of geometry at least since Euclid, the idea of symmetry as *self-coincidence* was not introduced until early in the 19th century. This was first done by A. L. Cauchy in an article on regular star-polyhedra. Meanwhile, crystallographers were defining the concepts of symmetry axes and planes in order to classify crystal forms; in 1849, A. Bravais discussed, in the spirit (but not in the language) of modern group theory, the motions of rotation, reflection and inversion which bring a polyhedron into coincidence with itself. The symmetry groups possible for polyhedra were eventually correctly enumerated after rotatory-reflection had been added to the list of possible isometries. As we mentioned earlier, [**1**] is a good reference for these groups.

The history of the theory of color symmetry provides an interesting illustration of the problem of finding a suitable mathematical interpretation of a concrete problem. The first investigators of color symmetry, concerned only with 2-color symmetry, focused attention on enumerating the 2-color groups of discrete motions of the sphere, the plane, and three-dimensional space [**21**]. This work was carried out without defining the more general problem. When attempts were made to generalize these ideas to polycolor symmetry, questions of admissibility and definition arose. To handle them, some authors ([**21**]) restricted the number of colors to 2, 3, 4, and 6, in accordance with the orders of rotation possible in crystal structures; others ([**13**]) required that $\phi(G)$ be cyclic. These and other requirements were imposed for mathematical convenience and ruled out many patterns of interest. (See [**18**] for a detailed discussion.)

Early in the development of color symmetry, the meaning of equivalence for color groups was intuitively understood to be the one we have given, but equivalence was never explicitly defined. The first explicit definition seems to have been given in [22]: two color groups G_H and $G_{H'}$ were defined to be equivalent if and only if H and H' are conjugate in G. As we have seen, this definition turned out to be too restrictive. The definition we have given seems to be widely, if implicitly, adopted now. A table of the k-color groups corresponding to the crystallographic and icosahedral point groups is given in [**7**].

The problem of enumerating colorings has only recently begun to be considered. It was first discussed by D. Harker [**7**], who proposed a classification much finer than the one presented here, which is based on that of R. Roth [**16**]. The colored spherical tessellations whose symmetry groups are transitive on the set of tiles, or edges, or vertices are enumerated in [**20**] by representing the groupings of labels by patterns on the sphere [**6**] and then enumerating the colorings of these patterns in the manner described in the last section. Other definitions of equivalence for colorings are possible and it is clear that this problem is still open.

It is also possible that the general requirement that the coloring be "perfect" may prove to be too restrictive for some applications. Imperfect or "partial" colorings are presently being studied from various points of view [**5**], [**16**]. Another obvious direction for generalization is to consider sets X on which the symmetry group G of the object does not act transitively (for example, the spherical tessellation representing the soccer ball). Here the problem is to decide what relation, if any, should exist between the colorings of the various orbits of G. Perfect color symmetry will probably turn out to be an important special case of a more general theory.

The theory of color symmetry, as described in this article, can be generalized to infinite discrete groups such as the crystallographic groups of the plane and of three-dimensional space, and their orbits; the checkerboard provides an example. Twenty years after the 2-color groups were enumerated, the 3-color plane groups were derived by Grünbaum [**3**]; the 3-color three-dimensional groups were recently derived by D. Harker [**9**]. Recent work on the k-color plane groups includes [**11**], which lists the k-color groups for $k \leqslant 15$; [**24**], which derives the k-color groups for $k \leqslant 60$; and [**19**], which enumerates certain infinite classes of k-color groups, including those having prime k, and which discusses the colorings admitted by the isohedral tilings of the plane.

This paper is dedicated to Caroline MacGillavry in recognition of her pioneering work in this field and in appreciation of her continued interest in it.

I am indebted to Caroline MacGillavry, Branko Grünbaum, and Lucia Krompart, Smith College '81, for their helpful comments on earlier versions of this paper. I would also like to thank Douglas Dunham, David Flesner and Rolph Schwarzenberger for their many valuable suggestions.

References

[1] H. S. M. Coxeter and W. O. J. Moser, Generators and Relations for Discrete Groups, 3rd ed., Springer-Verlag, 1972, sect. 4.2 and Table 2.

[2] E. H. Gombrich, The Sense of Order: A Study in the Psychology of Decorative Art, Cornell University Press, Ithaca, 1979.

[3] B. Grünbaum, Color symmetries and colored patterns, dittoed notes, University of Washington, Seattle, 1976. (See also [5]).

[4] B. Grünbaum and G. C. Shephard, Spherical tilings with transitivity properties, in In the Geometric Vein: Essays Presented to H. S. M. Coxeter, Springer-Verlag, Berlin, 1981.

[5] ______, Tilings and Patterns, W. H. Freeman, San Francisco, 1982.

[6] ______, Patterns on the 2-sphere, Mathematika, 28 (1981) 1–35.

[7] D. Harker, A table of the colored crystallographic and icosahedral point groups, Acta Cryst., A32 (1976) 133–139.

[8] ______, Colored lattices, in Structures of Matter and Patterns in Science, M. Senechal, ed., Schenkman, Cambridge, 1980.

[9] ______, Three-colored three-dimensional space groups, Acta Cryst., A37 (1981) 286–292.

[10] H. Heesch, Uber die vierdimensionalen Gruppen des dreidimensionalen Raumes, Z. Krist., 73 (1930) 325.

[11] J. Jarrett and R. Schwarzenberger, Coloured plane groups, Acta Cryst., A36 (1980) 884–888.

[12] O. Jones, Grammar of Ornament, Bernard Quatrich, London, 1868; reprinted Van Nostrand Reinhold, New York, 1972.

[13] A. Loeb, Color and Symmetry, Wiley-Interscience, New York, 1971.

[14] ______, Color symmetry and its significance for science, in Patterns of Symmetry, M. Senechal and G. Fleck, eds., University of Massachusetts Press, Amherst, 1977.

[15] C. MacGillavry, Symmetry Aspects of M. C. Escher's Periodic Drawings, International Union of Crystallography, Oosthoek, Utrecht, 1965. Reprinted as Fantasy and Symmetry, Abrams, New York, 1976.

[16] R. Roth, Color symmetry and group theory, Discrete Math., 38 (1982) 273–296.

[17] D. Schattschneider and W. Walker, M. C. Escher Kaleidocycles, Ballantine Books, New York, 1977.

[18] M. Senechal, Point groups and color symmetry, Z. Krist., 142 (1975) 1–23.

[19] ______, Color groups, Discrete Appl. Math., 1 (1979) 51–73.

[20] ______, Color symmetry and colored polyhedra, to appear in Acta Cryst.

[21] A. Shubnikov, N. Belov, and others, Colored Symmetry, translated from the Russian by J. Itzkoff and J. Golleb, W. Holser, ed., Pergamon, New York, 1964. (This book contains references to the original Russian articles.)

[22] B. L. van der Waerden and J. J. Burckhardt, Farbgruppen, Z. Krist., 115 (1961) 231–234.

[23] M. Wenninger, Spherical Models, Cambridge University Press, 1979.

[24] T. Wieting, The Mathematical Theory of Chromatic Plane Ornaments, Dekker, 1982.

So Many Things to Know

Does the zeta function have its zeros on the $\sigma = \frac{1}{2}$ line?
Is Fermat's theorem really true—Oh that would just be fine!

I cannot think of anything that would be much more fun
Than discovering another aleph between alephs null and one.

And I would be most thankful if only I could pry
A proof of the transcendence of e raised to the π.

One day toiling on such problems—'midst papers crumped and torn,
The phone rang and a voice barked, "Your account is overdrawn!"

—M. R. SPIEGEL

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References

[1] H. S. M. Coxeter and W. O. J. Moser, Generators and Relations for Discrete Groups, 3rd ed., Springer-Verlag, 1972, sect. 4.2 and Table 2.
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[5] ______, Tilings and Patterns, W. H. Freeman, San Francisco, 1982.
[6] ______, Patterns on the 2-sphere, Mathematika, 28 (1981) 1–35.
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[10] H. Heesch, Uber die vierdimensionalen Gruppen des dreidimensionalen Raumes, Z. Krist., 73 (1930) 325.
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[12] O. Jones, Grammar of Ornament, Bernard Quatrich, London, 1868; reprinted Van Nostrand Reinhold, New York, 1972.
[13] A. Loeb, Color and Symmetry, Wiley-Interscience, New York, 1971.
[14] ______, Color symmetry and its significance for science, in Patterns of Symmetry, M. Senechal and G. Fleck, eds., University of Massachusetts Press, Amherst, 1977.
[15] C. MacGillavry, Symmetry Aspects of M. C. Escher's Periodic Drawings, International Union of Crystallography, Oosthoek, Utrecht, 1965. Reprinted as Fantasy and Symmetry, Abrams, New York, 1976.
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[19] ______, Color groups, Discrete Appl. Math., 1 (1979) 51–73.
[20] ______, Color symmetry and colored polyhedra, to appear in Acta Cryst.
[21] A. Shubnikov, N. Belov, and others, Colored Symmetry, translated from the Russian by J. Itzkoff and J. Golleb, W. Holser, ed., Pergamon, New York, 1964. (This book contains references to the original Russian articles.)
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—M. R. Spiegel

NOTES

Formulas for Primes

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Formulas fascinate. Not everyone, of course, and not even every mathematician, but for some an array of symbols containing an equals sign has more allure and power than the same thing expressed in any other way. The reason for this probably sprouts in one's mathematical infancy, when we discover that manipulating symbols and using formulas can actually give us answers to problems. To the immature mathematical mind it seems almost magical; it seems as if we are getting something for nothing! First loves are not forgotten, so it is no surprise that a love of formulas, conscious or not, should persist into a person's mathematical adulthood.

Primes fascinate. Not everyone, of course, and not even every mathematician, but for some, contemplating their irregular but tantalizing not-quite-random march through the integers can lead to mystical experiences, or at least a desire to find some order in their chaos. Some victims of both of these fascinations look for formulas: formulas for p_n, **the nth prime**, or formulas for $\pi(n)$, **the number of primes less than or equal to n**, or formulas which give prime values exclusively. It is the purpose of this note to survey formulas for primes and show their wide variation from worthless, to interesting, to astonishing. I do not claim to include every formula ever found; in fact, a complete catalog would have little point. In weighing this representative sample, it seems that the balance tilts more to the side of worthless than astonishing. Although fascinating, a search for formulas for primes should be viewed in the same way as trying to find an elementary proof of Fermat's Last Theorem: good for recreation but almost certain to be fruitless.

Formulas for primes are not new, nor are they very old. Euler's curious polynomial $n^2 - n + 41$, which produces a prime for every integer n from 1 to 40, dates back to 1772. Not long after, Legendre and Gauss were trying to estimate $\pi(n)$, but it was not until the 1890s that authors started to publish formulas. They have, however, continued ever since. The first publications give formulas for χ, the **characteristic function for primes**, which is defined:

$$\chi(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{if } n \text{ is composite.} \end{cases}$$

An 1895 example is

$$\chi(n) = \frac{e^{2\pi i (n-1)!/n} - 1}{e^{-2\pi i/n} - 1}$$

[**D**, p. 427]. (For several reasons, some of our references are to secondary sources, hence we will use a slightly unorthodox method of noting these. References to volume I of Dickson's *History* [**1**] are noted by a **D** and those referring to items appearing in *Mathematical Reviews* are noted by an **MR**.)

Wilson's Theorem says that *n is prime if and only if* $(n-1)! \equiv -1$ *mod* n. This fact is the backbone of many formulas for functions which are cousin to $\chi(n)$. These functions have the property that they take on one type of value when n is prime and another type of value otherwise. A 1911 example [**D**, p. 428] is

$$f(n) = \sin^2 \pi n + \sin^2 \pi \left(\frac{1 + (n-1)!}{n} \right), n > 1; \tag{1}$$

$f(n)$ is zero if and only if n is prime. Another formula for the function $f(n)$ which does not depend on Wilson's Theorem is [**D**, p. 428]

$$f(n) = \frac{\sin^2 \pi n}{(\pi n)^2 (1 - n^2)^2} \cdot \sum_{k=2}^{\infty} \frac{\pi n}{k \sin \pi n / k}, n > 1.$$

Although Wilson's Theorem has proved to be notoriously unuseful in finding primes, this has not stopped the production of formulas in which it is essential. In a recent example, an author [**MR** 31(1966) #2198] takes the function

$$g(n) = \cos^2 \left(\pi \frac{(n-1)! + 1}{n} \right)$$

($g(n)$ is an integer if and only if n is prime) and uses this formula to produce a formula for p_{n+1}. Some other variations for the function $g(n)$ are [**16**],

$$g_1(n) = \frac{\sin^2(\pi (n-1)!/n)}{\sin^2(\pi/n)};$$

also [**MR** 45(1973) #8601]

$$g_2(n) = \frac{1 - \cos(\pi (n-1)!/n)}{1 + \cos(\pi/n)}$$

and [**MR** 21(1960) #7179]

$$g_3(n) = \frac{(j-1)!(n-j)! + (-1)^{j+1}}{n}.$$

Other ideas can be used too. Fermat's Theorem, which says that $a^{p-1} \equiv 1 \bmod p$ if p is prime and $(a, p) = 1$, is the idea in the formula [**MR** 40(1970) #4197]

$$\chi(n) = \prod_{2 \leqslant p \leqslant \sqrt{n}} \frac{1 - \cos(2n^{p-1}\pi/p)}{1 - \cos(2\pi/p)}$$

where $n \geqslant 4$ and the product is taken over primes. If n is composite, one of the primes will divide it, making the product zero; if n is prime, all of the factors in the product are 1. Another idea is to observe that $\lfloor n/i \rfloor < n/i$ for $i = 2, 3, \ldots, n-1$ if and only if n is prime (here $\lfloor x \rfloor$ denotes the greatest integer not greater than x). An author [**MR** 51(1976) #12676] used this to produce the formula

$$\chi(n) = \left\lfloor \left(\sum_{i=1}^{n-1} \lfloor \lfloor n/i \rfloor / (n/i) \rfloor \right)^{-1} \right\rfloor.$$

Once you have something like these, it is easy to play the formula game. For example, two authors, seventeen years apart [**D**, p. 432], [**D**, p. 434] noted that if in (1) you replace n by x, replace $(n-1)!$ by $\Gamma(x)$ and call the expression $f(x)$, then

$$\pi(n) = \frac{1}{2\pi i} \int_C \frac{f'(x)}{f(x)} dx$$

where C is a closed contour including the x-axis from 1 to n and excluding any complex zeros of f: a formula, but not one which has been of any computational use. As has been noted [**16**], once you have $\chi(n)$, you immediately have a formula for $\pi(n)$:

$$\pi(n) = \sum_{k=2}^{n} \chi(k). \tag{2}$$

If you also utilize C_n, the **characteristic function for the interval** $[0, n]$, that is, $C_n(k) = 1$ for $k \leqslant n$ and 0 for $k > n$, then you have a formula for p_{n+1}:

$$p_{n+1} = 2 + \sum_{k=2}^{\infty} C_n(\pi(k)). \tag{3}$$

The function C_n can be given an impressive analytic representation as an integral, or can be expressed by the combinatorial formula

$$C_n(k) = 1 + \sum_{j=0}^{k-n-1} (-1)^{j+1} \binom{n+j}{j} \binom{k}{n+1+j} \tag{4}$$

(where $\binom{s}{t} = 0$ if $t > s$). The function $\phi(n)$, which counts the number of positive integers less than n and relatively prime to n (called Euler's ϕ-function), and the floor function $\lfloor x \rfloor$ can be combined to give an equally impressive formula for $\pi(k)$:

$$\pi(k) = \sum_{i=2}^{k} \left\lfloor \frac{\phi(i)}{i-1} \right\rfloor. \tag{5}$$

The simple fact that $\phi(i) = i - 1$ if and only if i is prime makes (5) transparent. Now you can substitute (4) and (5) into (3) and there you are with a brand-new formula for p_{n+1}. Or you may take your favorite expression for $\chi(n)$, substitute it into (2) and obtain a new formula for $\pi(n)$. You may use these concoctions as your own, with no charge, to impress those who are impressed by such things.

The first formula for p_{n+1} appeared in 1900 [7]. It was based on the expression

$$F(n,k) = \frac{k!}{P(n,k)} + \frac{P(n,k)}{(k-1)!} - \left\lfloor \frac{(k-1)!}{P(n,k)} \right\rfloor \tag{6}$$

where

$$P(n,k) = 2^{e_1} 3^{e_2} 5^{e_3} \cdots p_n^{e_n}$$

where

$$e_i = \lfloor k/p_i \rfloor + \lfloor k/p_i^2 \rfloor + \lfloor k/p_i^3 \rfloor + \cdots .$$

For $2 \leqslant k < p_{n+1}$ it is well known that $P(n,k) = k!$, so (6) says that $F(n,k) = 1 + k - 0 = k + 1$. But when $k = p_{n+1}$, $P(n,k) = (k-1)!$ and so (6) says that $F(n,k) = k + 1 - 1 = k$. From these observations, with a little work, it is possible to get a formula which starts "$p_{n+1} = \cdots$." But the formula is a restatement of the fact that the prime-power decomposition of $k!$ differs from that of $(k-1)!$ by exactly one factor if and only if k is prime, and is just as useful for finding primes. The idea was not new, for another author [**D**, p. 437] had noted the previous year that p_{n+1} is the only solution greater than 1 of the equation

$$x! = x \prod_{i=1}^{n} p_i^{\lfloor x/p_i \rfloor + \lfloor x/p_i^2 \rfloor + \cdots}$$

I am not sure if G. H. Hardy was being satirical when he gave [**D**, p. 438] a formula for the largest prime dividing a positive integer x:

$$\lim_{r\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} \sum_{t=0}^{m} \left(1 - \left(\cos\left[(t!)^r \pi/x\right]\right)^{2n}\right).$$

It takes more than a little effort to understand the computationally useless formula, but none to understand the words. In any event, the joke, if a joke it was, did not kill the genre. The author of a recent example [**MR** 52(1976) #246, where the formula is misquoted] defines (using Wilson's Theorem again)

$$f(n) = \operatorname{sgn}\left(\frac{2(n-1)!}{n} - \left\lfloor \frac{2(n-1)!}{n} \right\rfloor\right);$$

here $\operatorname{sgn}(x) = -1, 0$, or 1 for x negative, zero, or positive respectively. This is an opaque way of writing "$f(n) = 1$ if n is an odd prime and 0 otherwise." He then writes that for $n \geqslant 2$

$$\begin{aligned} p_{n+1} = {} & (p_n + 2)f(p_n + 2) + (p_n + 4)f(p_n + 4)(1 - f(p_n + 2)) \\ & + (p_n + 6)f(p_n + 6)(1 - f(p_n + 2))(1 - f(p_n + 4)) \\ & + (p_n + 8)f(p_n + 8)(1 - f(p_n + 2))(1 - f(p_n + 4))(1 - f(p_n + 6)) \\ & + \cdots . \end{aligned}$$

Notice that all of the terms except one on the right are zero. Since there is no chance of evaluating $f(n)$ other than by knowing when n is prime, the formula amounts to no more than the statement that p_{n+1} is the first prime after p_n. As H. S. Wilf has recently observed in a similar context [**15**], this latter statement is unacceptable in polite society. That the formula appeared in a respected journal is testimony to the power of formulas. A similar result appeared in 1950 [**MR** 12(1951) #392]:

$$\pi(n) = n - \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=1}^{n} \cos(xP(m))\, dx \tag{7}$$

where

$$P(m) = \prod_{j=1}^{m-1} \prod_{k=1}^{m-1} (jk - m)$$

and $P(1) = 0$. For $m \geqslant 2$, $P(m) = 0$ exactly when m is composite, so the second term on the right of (7) is just the number of composite integers not greater than n. Unless some other way is found to compute $P(m)$ (very doubtful) the formula is a truism. Another author (cited in [**14**, p. 149]) gave a formula somewhat reminiscent of Hardy's: for $n \geqslant 3$,

$$\pi(n) = 1 + \sum_{k=3}^{n} \left(1 - \lim_{m \to \infty} \left(1 - \prod_{j=2}^{k-1} \sin^2(k\pi/j)\right)^m\right).$$

I leave to the reader the task of deciphering why the formulas work, and the task of computing with them.

Here is the other extreme. W. H. Mills [**11**] proved that *there is a real number A such that $\lfloor A^{3^n} \rfloor$ is prime for* **all** n, $n = 1, 2, 3, \ldots$. Isn't that astonishing? Doesn't it make you wonder how it could be true? The proof shows that we do not have an infinite prime-generator because we cannot find A by any means other than constructing it, and to construct it we need to be able to recognize arbitrarily large primes. The result was so striking that it provoked a large number of papers [**MR** 11(1950) #664], [**MR** 13(1952) #321], [**MR** 13(1952) #321a], [**MR** 14(1953) #256], [**MR** 15(1954) #11], including a proof by E. M. Wright [**17**] that there are infinitely many suitable numbers A and moreover, the set of all of them has cardinality c, measure 0, and is nowhere dense. A satisfying result; we now know the odds of finding A. A summary of the above activity can be found in [**2**].

The following idea was discovered independently at about the same time by two authors [**MR** 14(1953) #355], [**MR** 14(1953) #621]. Let

$$s = 0.2003000050000000700000000110\ldots = \sum_{n=1}^{\infty} p_n/10^{n^2}.$$

The real number s contains all the primes separated by many zeros, and the formula

$$p_n = \lfloor 10^{n^2} s \rfloor - 10^{2n-1} \lfloor 10^{(n-1)^2} s \rfloor$$

retrieves them. Neither author was aware that Leo Moser had done the same thing earlier [**12**] using the number $\sum_{n=1}^{\infty} p_n/10^{n(n+1)/2}$ instead, calling his own work "admittedly rather trivial." Trivial or not, both results were later generalized [**MR** 24(1962) #A1869]. The idea of putting the primes into a real number so as to get them out later could have come from Mills' theorem.

I suspect that the late J. M. Gandhi thought that his formula for primes [**3**] actually had some chance of being useful. He proved that if $P_n = p_1 p_2 \cdots p_n$, then p_{n+1} is the unique integer m satisfying the inequality

$$1 < 2^m\left(\sum_{d|P_n} \mu(d)/(2^d - 1) - 1/2\right) < 2 \tag{8}$$

(here μ is the Mobius function: $\mu(1) = 1$ and for $d > 1$, $\mu(d) = 0$ if d has a square factor; if not, $\mu(d) = 1$ if d has an even number of prime divisors and $\mu(d) = -1$ otherwise). The sum in (8) is finite, it involves no primes larger than p_n, and perhaps some way of evaluating it easily could be found. But S. W. Golomb later showed [**4**] that the formula is a version of the Sieve of Eratosthenes. Another hope dashed! Golomb [**5**] used similar ideas to get other formulas, such as

$$p_{n+1} = \lim_{s \to 0} \left(P_n(s)\zeta(s) - 1\right)^{-1/s}$$

where

$$P_n(s) = \prod_{i=1}^{n} (1 - p_i^{-s}) \text{ and } \zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Other formulas based on the Sieve had appeared earlier ([**MR** 15(1954) #685], [**MR** 26(1963) #1289], and [**MR** 27(1964) #101] as was noted in [**13**]). They are still coming [**MR** 81j #10008].

Though not a formula, the following result of Mann and Shanks [**10**] is remarkable. Write Pascal's triangle with row k starting in column $2k$:

	Column 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Row 0	1														
1		1	1												
2					1	2	1								
3							1	3	3	1					
4									1	4	6	4	1		
5											1	5	10	10	5
6													1	6	15
7															1 ....

They proved that in this array: *a column number is prime if and only if each number in it is divisible by the corresponding row number*. Is that not charming? H. W. Gould later showed [**6**] that the result is equivalent to the statement: n is prime if and only if k divides $\binom{k}{n-2k}$ for all $k > 1$ such that $n/3 \leqslant k \leqslant n/2$. Another pleasing result [**9**] is a suggested algorithm to find the next prime after p: add to p the smallest positive integer i not of the form $i\lfloor (p+i)/i \rfloor - p$ for $i = 2, 3, \ldots, p$. This works assuming the truth of the almost surely true but unproven statement that there is a prime between n and $n + n^{1/2}$ for n sufficiently large.

Deep results on Diophantine sets resulting from work on Hilbert's Tenth Problem have made it possible to prove that the set of primes is exactly the set of positive values of some polynomial. In [**8**] such a polynomial, of degree 25 in 26 variables, is actually written out. That too is startling; you might wonder why no computer has been set to work substituting numbers in it, since every time a value is positive it must be prime. The reason is that the form of the polynomial is

$$(x_{11} + 2)\left(1 - \sum_{i=1}^{14} \left(P_i(x_1, x_2, \ldots, x_{26})\right)^2\right)$$

so it is positive only if fourteen rather complicated polynomials simultaneously vanish. Hence the first positive value might not appear until considerably after the end of the universe, and even then it might be something trivial, like 17.

The conclusion to be drawn from all this, I think, is that formulas for formulas' sake do not advance the mathematical enterprise. Formulas should be useful. If not, they should be astounding, elegant, enlightening, simple, or have some other redeeming value. (Conway's prime-producing machine, this MAGAZINE, pp. 26-33, succeeds on all counts.) Authors who discover formulas should not rush into print. Even as in business and marriage, in mathematics not *all* that is true needs to be published. Gauss, as always, had it right: *pauca sed matura*.

The author wishes to express awe at the magnitude of the referee's efforts and gratitude for the many helpful suggestions.

References

[1] L. E. Dickson, History of the Theory of Numbers, Chelsea, New York, 1952; reprint of the 1919 edition.

[2] U. Dudley, History of a formula for primes, Amer. Math. Monthly, 76(1969) 23–28. MR 38(1969) #4270.

[3] J. M. Gandhi, Formulae for the *n*th prime, Proc. Washington State Univ. Conf. on Number Theory, Washington State Univ., 1971, pp. 96–106. MR 48(1974) #218.

[4] S. W. Golomb, A direct interpretation of Gandhi's formula, Amer. Math. Monthly, 81(1974) 752–754. MR 50(1975) #7003.

[5] ______, Formulas for the next prime, Pacific J. Math., 63(1976) 401–404. MR 53(1977) #13094.

[6] H. W. Gould, A new primality criterion of Mann and Shanks, Fibonacci Quart., 10(1972) 355–364, 372. MR 47(1974) #119.

[7] C. Isenkrahe, Ueber eine Losung der Aufgabe, jede Primzahl als function der vorhergehenden Primzahlen durch einen geschlossenen Ausdruck darzustellen, Math. Ann., 53(1900) 42–44.

[8] J. P. Jones, D. Sato, H. Wada, D. Wiens, Diophantine representation of the set of prime numbers, Amer. Math. Monthly, 83(1976) 449–464. MR 54(1977) #2615.

[9] Steven Kahan, On the smallest prime greater than a given positive integer, this MAGAZINE, 47(1974) 91–93. MR 48(1974) #10964.

[10] H. B. Mann and Daniel Shanks, A necessary and sufficient condition for primality and its source, J. Combin. Theory Ser. A., 13(1972) 131–134. MR 46(1973) #5225.

[11] W. H. Mills, A prime representing function, Bull. Amer. Math. Soc., 53(1947) 604. MR 8(1947) #567.

[12] Leo Moser, A prime representing function, this MAGAZINE, 23(1950) 163–164.

[13] K. S. S. Namboordiripad, A note on formulae for the *n*th prime, Monatsh. Math., 75(1971) 256–262. MR 46(1973) #126.

[14] Joe Roberts, Elementary Number Theory, MIT Press, Cambridge, 1977. MR 58(1979) #16472.

[15] H. S. Wilf, What is an answer?, Amer. Math. Monthly, 89(1982) 289–292.

[16] C. P. Willans, On formulae for the *n*th prime number, Math. Gaz., 48(1964) 413–415. MR 31(1966) #2198.

[17] E. M. Wright, A class of representing functions, J. London Math. Soc., 29(1954) 63–71. MR 15(1954) #288.

***Prime* observation 1. The forest *prime*val.**

Polynomial Translation Groups

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Consider the 4×4 matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & 2r & 1 & 0 \\ r^3 & 3r^2 & 3r & 1 \end{bmatrix}, r \text{ a scalar.}$$

Is this set of matrices closed under multiplication? Inversion? Do two such matrices commute? (You have probably answered "yes" to each question on the basis of gamesmanship. If so, it is probable that you also view these conclusions as unexpected or striking. Hence, it would have been more to the point to begin "Isn't it interesting that this set is closed under multiplication," etc. But then I couldn't have written ...). Yes, yes, and yes. Let me show you a reason why.

If $i \leqslant j$, the ij entry of the matrix above is given by

$$m_{ij}(r) = \binom{i-1}{j-1} r^{i-j}, \tag{1}$$

that is, the ith row contains the terms of the binomial expansion of $(r+1)^{i-1}$. As there is no particular virtue in considering 4×4 matrices, let n be a fixed positive integer, and define for each number r a lower triangular $n \times n$ matrix $M(r)$ according to (1). Then

$$M(r)M(s) = M(r+s), \tag{2}$$

so that the set of matrices $M(r)$ is a multiplicative group.

Arm yourself from your favorite arsenal of combinatorial identities and you are sure to emerge from battle with (2) the victor. However, once I convince you that geometrically $M(r)$ is a translation of r units, you will perceive the validity of (2) without recourse to such weaponry. On the contrary, (2) will yield up an identity or two to add to the arsenal.

Every translation requires two things: something to move and a space to move it in. The translations of interest here move polynomials about in the Cartesian plane. Consider the space of polynomials $p(x)$ of degree less than n, and identify each polynomial with its graph. Each polynomial may be represented by a vector of coefficients with respect to the basis $\mathbf{B} = \{1, x, x^2, \ldots, x^{n-1}\}$, and linear transformations can then be represented by matrices. In particular, let the transformation T_r move a polynomial r units to the left. That is, if p is a polynomial, $T_r p$ is the polynomial defined by

$$T_r p(x) = p(x+r). \tag{3}$$

By evaluating T_r at each element of $\mathbf{B}$, and expressing the results as coefficient vectors, we derive the rows of $M(r)$. Thus, if the coefficient vector of p is $\mathbf{v}$ and that of $T_r p$ is $\mathbf{w}$, then $\mathbf{w} = \mathbf{v}M(r)$; $M(r)$ represents a translation of r units to the left. And, since composing translations of r units and s units yields a translation of $r+s$ units, (2) is established. Moreover, by (2), the mapping $r \to M(r)$ establishes an isomorphism from the additive group of the real numbers into the multiplicative structure of the $n \times n$ matrices. The image of this mapping is consequently a commutative group. This result applies more generally to polynomials and matrices over an arbitrary field if translations are defined as in (3).

Having established (2) by exploiting a fortuitous geometric interpretation, it is rewarding to consider the toil avoided. To derive (2) directly, it is necessary to prove

$$\sum_{k=1}^{n} m_{ik}(r) m_{kj}(s) = m_{ij}(r+s),$$

which, after using (1) and omitting the zero terms from the sum, becomes

$$\sum_{k=j}^{i}\binom{i-1}{k-1}r^{i-k}\binom{k-1}{j-1}s^{k-j}=\binom{i-1}{j-1}(r+s)^{i-j};\ 1\leqslant j\leqslant i\leqslant n. \tag{4}$$

Luckily, we already know (2) is true and so we get (4) gratis, avoiding the effort of a direct derivation. Moreover, other identities can be deduced easily from (4). For example, replacing i and j by $I+1$ and $J+1$, respectively, eliminating the irrelevant reference to n, and reindexing the sum to run from J to I gives

$$\sum_{k=J}^{I}\binom{I}{k}\binom{k}{J}r^{I-k}s^{k-J}=\binom{I}{J}(r+s)^{I-J};\ 0\leqslant J\leqslant I. \tag{5}$$

This identity may be viewed as a generalization of the binomial theorem (to which it reduces when $J=0$). As special cases, take $r=1$ and $s=1$ or -1 to derive

$$\binom{I}{J}2^{I-J}=\sum_{k=J}^{I}\binom{I}{k}\binom{k}{J}\text{ and }\sum_{k=J}^{I}(-1)^k\binom{I}{k}\binom{k}{J}=0.$$

The second of these identities appears as entry 3.119 in [**1**]. Similarly, when $r=1$, identity (5) is essentially the same as entry 3.118 of [**1**], which is a normalized version of (5).

We have shown that equation (5) has interpretations in terms of geometry and matrix algebra. There is also a combinatorial interpretation. Imagine a painter who is to paint strips of paper using $r+s$ colors, in the following way. Each strip is divided into I sections. Of these, J are to remain blank, the remaining $I-J$ must be painted using any color of the $r+s$, one color for each section. The painter may produce, in this fashion, $\binom{I}{J}(r+s)^{I-J}$ differently painted strips. Alternatively, let the painter first choose some number k of the I sections (with k no less than J) and paint the remaining $I-k$ sections using the first r colors. Now k sections are blank. The painter selects J of these to leave blank, and paints the other $k-J$ using s colors. Following these steps, there are

$$\sum_{k=J}^{I}\binom{I}{k}r^{I-k}\binom{k}{J}s^{k-J}$$

different ways to paint a strip. In following either set of instructions, the artist can produce any painted strip with J blank parts, so the formulas enumerating the set of outcomes must agree, which verifies (5).

Buried in (5) somewhere is the combinatorial equivalent of $M(r)M(-r)=I$. The geometric justness of this result is evident but the explicit matrix formulation is a pleasant surprise. To invert $M(r)$, simply change the sign of every other entry. For example, the inverse of the 4×4 matrix displayed at the beginning of this note is

$$\begin{bmatrix}1&0&0&0\\-r&1&0&0\\r^2&-2r&1&0\\-r^3&3r^2&-3r&1\end{bmatrix}.$$

As a final topic, I'd like to point out some ways to exploit this example. Note first that replacing r with a numerical value disguises the pattern of the matrix. Thus, given the matrices

$$\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&4&1&0\\8&12&6&1\end{bmatrix},\quad\begin{bmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{bmatrix},\text{ and }\begin{bmatrix}1&0&0&0\\3&1&0&0\\9&6&1&0\\27&27&9&1\end{bmatrix}$$

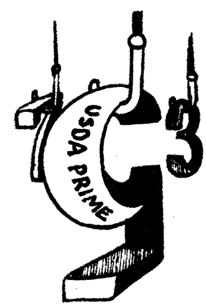
as $M(2)$, $M(-1)$, and $M(3)$, respectively, a student might find it challenging to determine the general form, $M(r)$. In a similar vein, as a source of matrices with known inverses, the translation matrices are useful for concocting matrix inversion exercises. Students may find a certain amount of charm in setting out to invert a 4×4 matrix and ending up (nearly) back at the original matrix.

More abstract problems can be designed to exercise a student's understanding of axioms. For instance, having a student check the set of 3×3 $M(r)$'s for closure under multiplication provides practice with axioms, and in particular, that troublesome notion of closure. Posing similar problems without specifying n adds another increment in difficulty. Finally, the example may be presented much as it is here. The specific lesson that an appropriate geometric interpretation reveals algebraic structure and combinatorial identities is a valuable one. More generally, this example may hint at the fertility that results from multiple interpretations of a single mathematical structure through the use of isomorphisms.

Reference

[1] Henry W. Gould, Combinatorial Identities, Morgantown Printing and Binding, Morgantown, West Virginia, 1972.

In his *prime*



USDA *prime*

The *prime* coat

Prime observations 2.

Conway's Prime Producing Machine

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Conway does it again! He's already given us Life and Sprouts, Phutball and Hackenbush, the Doomsday Rule and Sylver Coinage, and dozens of other things [**1**], bewildering on first acquaintance, but enticingly arranged and punctuated with pedagogy so that we can't help learning about them.

It's an old adage that you don't really understand something until you teach it to someone else. Donald Knuth extends this to teaching it to a computer. John Horton Conway is never satisfied with his exposition until he can explain his latest interest to the person-in-the-street, even one without mathematical training.

If you understand logic and computers all the way from Turing machines to the implementing of a program that you've written in a high-level language, then this article isn't for you. But before you go, you must at least be intrigued by Conway's machine, a row of fourteen rational numbers (Figure 1).

$\frac{17}{91}$	$\frac{78}{85}$	$\frac{19}{51}$	$\frac{23}{38}$	$\frac{29}{33}$	$\frac{77}{29}$	$\frac{95}{23}$	$\frac{77}{19}$	$\frac{1}{17}$	$\frac{11}{13}$	$\frac{13}{11}$	$\frac{15}{14}$	$\frac{15}{2}$	$\frac{55}{1}$
A	*B*	*D*	*H*	*E*	*F*	*I*	*R*	*P*	*S*	*T*	*L*	*M*	*N*

Figure 1. **John Horton Conway's Prime Producing Machine.**

The letters in Figure 1 were originally the first fourteen in the alphabet, but we've changed some of them to make them more mnemonic. For example, *N* introduces the Next Number and *P* Prints its Primeness. The **input** is the number 2. A **step** is to multiply the current number by the *earliest* member of the row in Figure 1 which gives a *whole number* answer. **Output** happens whenever a pure power of 2 occurs. For example, the first nineteen steps (Figure 2) lead to 2^2 as the first power of 2, and the exponent 2 is the first prime. If you're patient enough to carry out fifty more steps, and don't make any mistakes, you'll get to 2^3 with 3 as the second prime. A mere 211 steps later 2^5 appears, giving 5 as the third prime, and so on.

2	*M*	15	*N*	825	*E*	725	*F*	1925	*T*	2275	*A*	425	*B*	390	*S*	330
					E	290	*F*	770	*T*	910	*A*	170	*B*	156	*S*	132
					E	116	*F*	308	*T*	364	*A*	68	*P*	4		

Figure 2. **The First Nineteen Steps.**

What is going on? Before we answer that, let's write our own program to produce primes, keeping it as simple as we can (Figure 3).

Perhaps that's too naive? How can we tell if n is prime? Its only positive divisors must be n and 1. So a simple, albeit slow, procedure would be to divide n by $n-1, n-2, \cdots, 2, 1$ in turn. If we find a divisor *before* we get to 1, then n is **composite**. If we have to go all the way to 1, then n is **prime** (Figure 4).

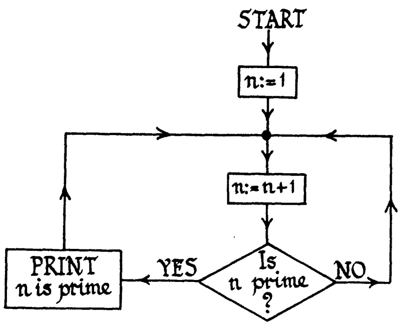


FIGURE 3. **A Program to Produce Primes.**

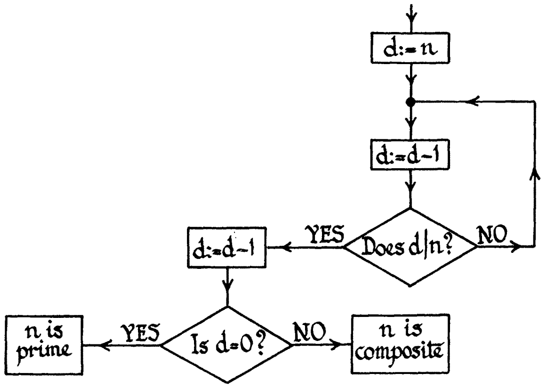


FIGURE 4. **Is n Prime?**

How do we answer the question "does d divide n?"? We want to write $n = dl + r$ where $0 \leqslant r < d$. Then, if $r = 0$, $d|n$, and if not, then $d \nmid n$. Since we're only interested in r, we can replace division by subtraction, decreasing n by decrements of d until the remainder is small enough (FIGURE 5). The loop on the right is performed l times.

There used to be an admission examination, the "11-plus", to British secondary schools. A question that was asked on one occasion was "Take 7 from 93 as many times as you can." One child answered, "I get 86 every time." I hope she got her place!

In FIGURE 5 there are still two boxes which contain ideas that are too sophisticated for the very simple computer that we have in mind: the subtraction, $r := r - d$, and the comparison, is $r < d$? Our computer will only do the following three things:

(a) add one to the contents of a register $(+1)$,
(b) belittle the contents by one (-1), and
(c) see if a register is empty (0).

To solve the problems of simplifying subtraction and comparison, let's get some hints by taking a longer look at Conway's machine. Rewrite the row of rationals from FIGURE 1 in factored form (FIGURE 6).

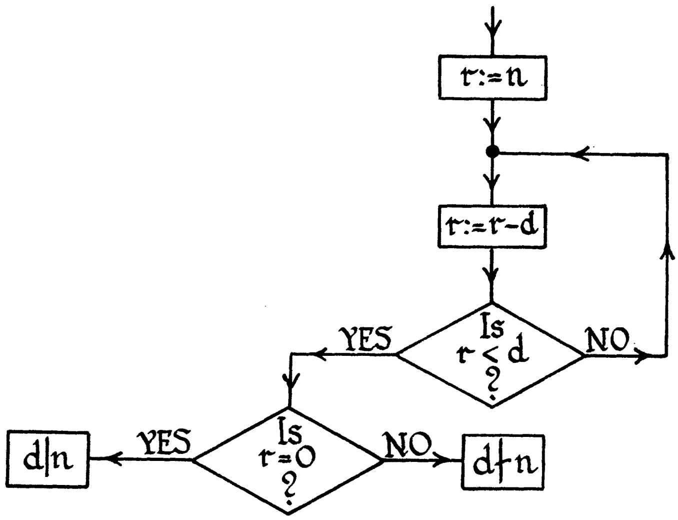


FIGURE 5. **Deciding if d Divides n.**

$\frac{17}{7\cdot 13}$	$\frac{2\cdot 3\cdot 13}{5\cdot 17}$	$\frac{19}{3\cdot 17}$	$\frac{23}{2\cdot 19}$	$\frac{29}{3\cdot 11}$	$\frac{7\cdot 11}{29}$	$\frac{5\cdot 19}{23}$	$\frac{7\cdot 11}{19}$	$\frac{1}{17}$	$\frac{11}{13}$	$\frac{13}{11}$	$\frac{3\cdot 5}{2\cdot 7}$	$\frac{3\cdot 5}{2}$	$\frac{5\cdot 11}{1}$
A	B	D	H	E	F	I	R	P	S	T	L	M	N

FIGURE 6. **Conway's Machine in Factored Form.**

Notice that after any step the current number is always in the shape

$$2^t 3^s 5^r 7^q p \tag{1}$$

where t, s, r, q are non-negative integers, the **contents** of the **registers**, represented by the four **small primes**, 2, 3, 5, 7; and p is either 1 or one of the six **big primes**, 11, 13, 17, 19, 23 or 29. These big primes and 1 are the seven **states** of the machine. They are depicted in FIGURE 7, in which the fourteen directed arcs are the **transitions**, corresponding to the fourteen fractions, each labeled with its appropriate letter, and, if the contents of the registers are affected, with a multiplier indicating this. For example, step B, which takes the machine from state 17 to state 13, has the multiplier $2\cdot 3/5$, which increases t and s by one, decreases r by one, and leaves the 7-register unchanged.

We can short-circuit parts of the state diagram and speed up our calculations by noticing that some sequences of steps naturally amalgamate into **subroutines**. For example, when in state 19, the machine will make the pair of steps HI (H Hacks away at the 2-register, while I Increases the 5-register correspondingly) t times, where t is the contents of the 2-register, transferring t to the 5-register, and then do step R (R Replaces one in the 7-register to compensate for D's Diminishing of the 3-register), arriving at state 11. Since state 19 is reached only by step D from state 17, or by step I from state 23, we can replace states 19 and 23 by the subroutine

$$D_n = D(HI)^n R$$

which goes Directly from state 17 to state 11.

Similarly we can eliminate state 29 by the subroutine

$$T_s = (EF)^s T.$$

From state Eleven, E Empties the contents, s, of the 3-register while F Feeds them into the 7-register; then T Takes us to state Thirteen. These and other subroutines are listed in TABLE 1, where the vector $(t, s, r, q)_p$ represents the number (1), i.e., it gives the contents of the 2-, 3-, 5- and 7-registers and the state p as a subscript.

In the Next Number subroutine, $N_n = L^{d-1}M^{n-d+1}N$, steps L and M Lessen the contents of the 7- and 2-registers to zero and Make up those of the 3- and 5-registers to n. Then N goes to the Next Number by adding 1 to the 5-register.

Subroutine	from	to	Description
$T_s = (EF)^s T$	$(t, s, r, q)_{11}$	$(t, 0, r, q+s)_{13}$	Transfers contents, s, of Three-register to 7-register.
$S_d = (AB)^d S$	$(t, s, r, d)_{13}$	$(t+d, s+d, r-d, 0)_{11}$	Subtracts d from r and Sums it to s and t.
$A_r = (AB)^r A$	$(dl, 0, r, d)_{13}$	$(n, r, 0, d-r-1)_{17}$	Asks: Is $r = 0$? Anything in 5-register?
$D_n = D(HI)^n R$	$(n, r, 0, d-r-1)_{17}$	$(0, r-1, n, d-r)_{11}$	Decreases Divisor by one.
$N_n = L^{d-1}M^{n-d+1}N$	$(n, 0, 0, d-1)_1$	$(0, n, n+1, 0)_{11}$	Next Number, $n := n+1$.

TABLE 1. **Subroutines.**

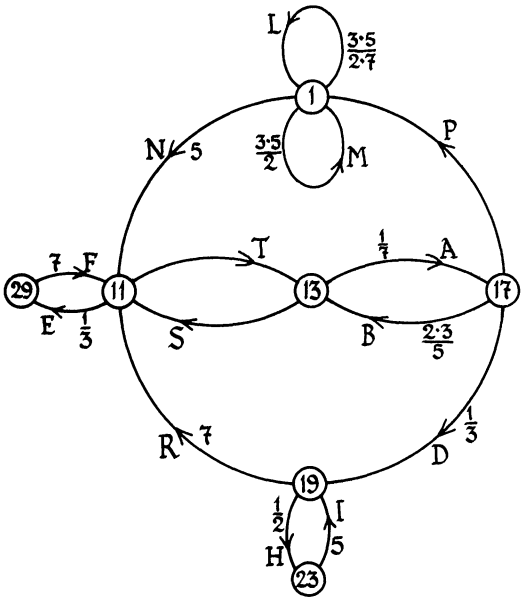


FIGURE 7. **The Seven States and Fourteen Transitions.**

Here's an analysis of states 13 and 17. These are the states where the questions "Is $q = 0$?", "Is $r = 0$?", "Is $s = 0$?" are answered.

13	A	Asks About the contents of the 7-register, q: **if** $q > 0$, $q := q - 1$ And goes to 17; **else** $q = 0$, S Shifts to State 11.
17	B	Brings you Back to 13 **if** the 5-register, r, is positive; **else** $r = 0$ and **if** $s > 0$
	D_n	Decreases the Divisor by one and goes to 11; **else** $s = 0$ and
	P	Prints the Primeness (composite if $d - 1 > 0$) of n and goes to 1.

FIGURE 8 is a condensation of FIGURE 7, using the subroutines of TABLE 1.

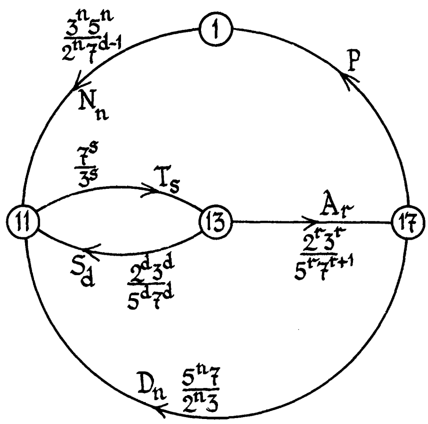


FIGURE 8. **A Condensed State Diagram.**

	2-	3-	5-	7-		2-	3-	5-	7-		2-	3-	5-	7-
	1				N_3		3	4		T_1			5	2
N_1		1	2		T_3			4	3	S_2	2	2	3	
T_1			2	1	S_3	3	3	1		T_2	2		3	2
S_1	1	1	1		T_3	3		1	3	S_2	4	2	1	
T_1	1		1	1	A_1	4	1		1	T_2	4		1	2
S_1	2	1			D_4			4	2	A_1	5	1		
T_1	2			1	T_0			4	2	D_5			5	1
A_0	2				S_2	2	2	2		T_0			5	1
P	2	is prime			T_2	2		2	2	S_1	1	1	4	
N_2		2	3		S_2	4	2			T_1	1		4	1
T_2			3	2	T_2	4			2	S_1	2	1	3	
S_2	2	2	1		A_0	4			1	T_1	2		3	1
T_2	2		1	2	P	4	has a factor		1 + 1	S_1	3	1	2	
A_1	3	1			N_4		4	5		T_1	3		2	1
D_3			3	1	T_4			5	4	S_1	4	1	1	
T_0			3	1	S_4	4	4	1		T_1	4		1	1
S_1	1	1	2		T_4	4		1	4	S_1	5	1		
T_1	1		2	1	A_1	5	1		2	T_1	5			1
S_1	2	1	1		D_5			5	3	A_0	5			
T_1	2		1	1	T_0			5	3	P	5	is prime		
S_1	3	1			S_3	3	3	2						
T_1	3			1	T_3	3		2	3					
A_0	3				A_2	5	2							
P	3	is prime			D_5		1	5	1					

TABLE 2. **The First 280 Steps.**

In TABLE 2 we use the subroutines to compress the calculation of the first 280 steps.

If you've tried working the machine yourself, you'll already have noticed a number of further coagulations of subroutines into routines, and other patterns.

1. The sum of the contents of the 2- and 5-registers, $t+r$, is always equal to n (though fleetingly it's one less in state 23, which we've eliminated). We need to store n, but we also need to decrease it by decrements of d, and to examine the remainder. Conway does this economically by storing n as the *sum* of *two* registers: think of t as the *total* so far subtracted from n, and r as the *remainder*.

2. In the same way the sum of the contents of the 3- and 7-registers, $s+q=d$ (except momentarily in state 29, which we've also eliminated). Think of s as the *subtrahend*, the part of d that has been subtracted, and q as *quantus*, how much of d is left.

3. The Next Number subroutine, N_n, is always followed by T_n (note that this is the *old* n in process of becoming $n+1$) and the Decrease Divisor subroutine, D_n, is always followed by T_{r-1}, and each of these is followed by the **division routine**, $(S_d T_d)^l A_r$.

Now we can fill in the missing "subtraction and comparison" bits of our program (FIGURE 9). In the figure, the loop marked EF will be performed s times on each occasion that it's entered, the loop ST will be performed l times, and the loop AB m times, where $m=\min(d, r)$, so that AB is performed $dl+r=n$ times altogether.

Finally, let's fit FIGURES 3, 4, 5 and 9 together in a single diagram (FIGURE 10) but to keep it from getting too big, we've used some of the subroutines to condense it a bit. So that you can follow along, we've decorated it with state numbers and step labels. For homework, to make quite sure that you know what's going on, write out the whole program in the usual way.

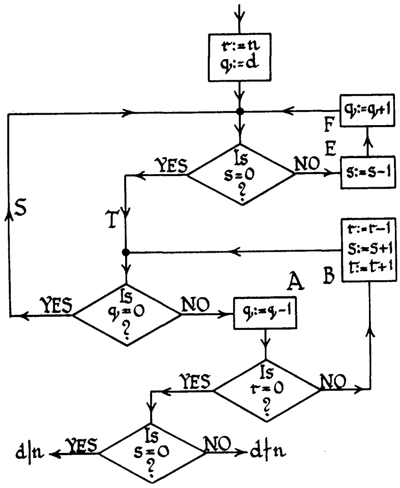


FIGURE 9. **The Division Routine: Subtraction and Comparison.**

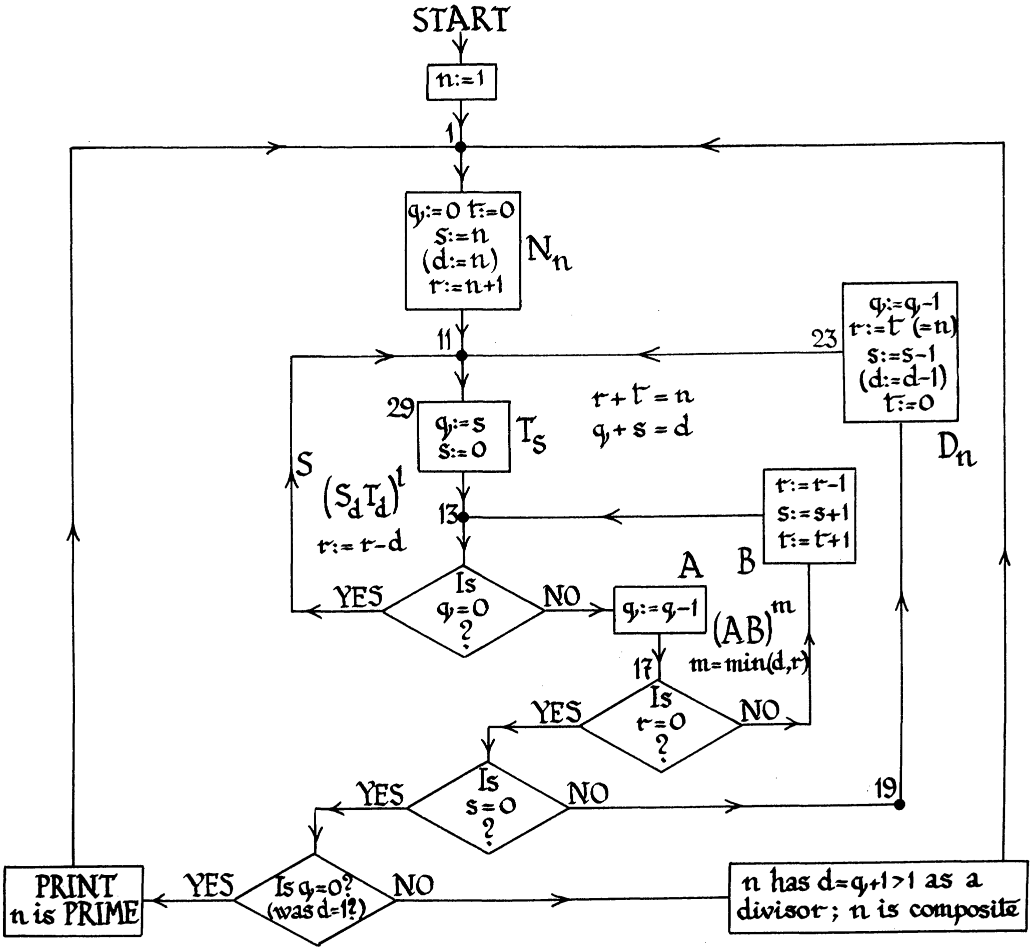
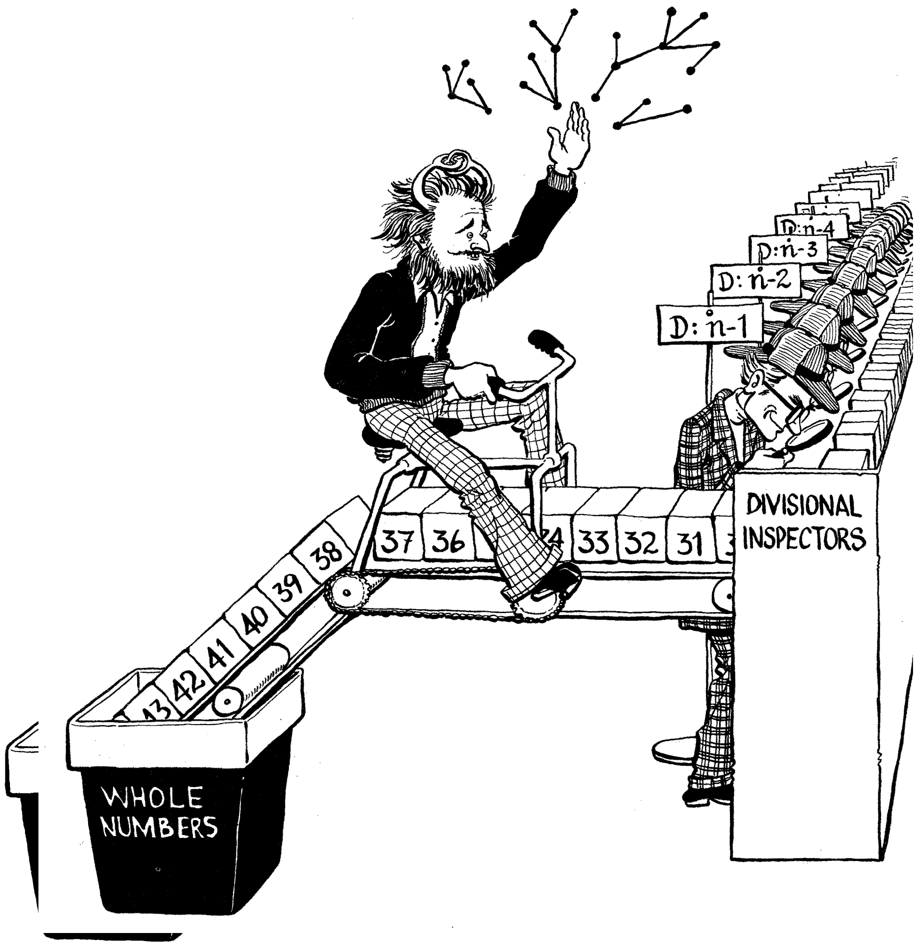


FIGURE 10. **A Prime Printing Program.**

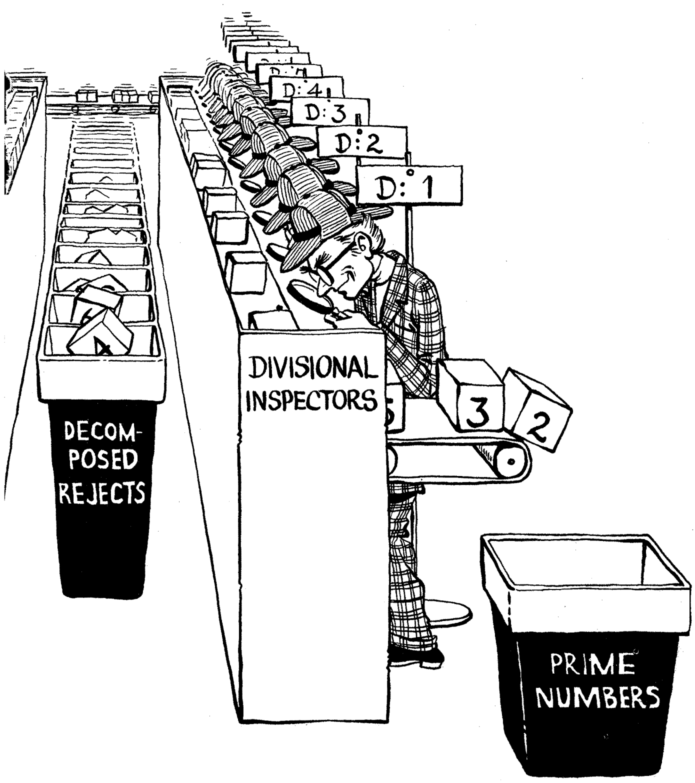


One of the first prime producing machines was the sieve of Eratosthenes. There's a picture of another (essentially due to Euclid!) in [**5**] but the inside must be a much cleverer set-up than that depicted in our cartoon above. Conway's machine appeared as a problem in [**2**]. There was a suggestion [**6**] that $R = 77/19$ and $N = 55/1$ should be replaced by $11/19$ and $275/1$. This would certainly speed things up, but speed is not quite what we're aiming for, and the introduction of a 3-digit number and the omission of the oddest prime, 2, must be regarded as blemishes. Even more important is that the suggested modification would no longer qualify as a **Minsky machine** [**4**], performing only the three simple operations (a), (b) and (c). Because $275 = 5^2 11$, the new N step would add *two* to the 5-register. One of the main messages of the machine is the immense gap between theory and practice: you spoil the point if you start to try to bridge that abyss. A similar example is the computer you can build using Conway's game of Life [**1**, chap. 25]. It's easy to visualize this as being programmed to produce primes, but it's hardly a practical proposition!

Postscript. In [**2**] Conway asked how long it would take to generate the first thousand primes in this way. Here's some information to help you answer this. The thousandth prime is 8831. The number of steps needed to inspect the number n is

$$n - 1 + (6n + 2)(n - b) + 2 \sum_{d=b}^{n-1} \lfloor n/d \rfloor$$

where b is the biggest divisor of n apart from n itself and $\lfloor\ \rfloor$ is Donald Knuth's **floor** symbol,



meaning "greatest integer not greater than." For example, even numbers ($b = n/2$) take $3n^2 + 3n + 1$ steps and odd multiples of three ($b = n/3$) take $4n^2 + 4n + 2$ steps, while primes ($b = 1$) take

$$3(n-1)(2n+1) + 2\sum_{d=1}^{n-1} \lfloor n/d \rfloor$$

steps, where Σ is a little more than $n \ln n$. For

$n=2$	3	5	7	11	13	17	...	8831
$\Sigma=2$	4	9	15	28	36	51	...	81591.

Just to inspect 8831 alone takes 468056052 steps; Omar [**1**] will no doubt calculate the exact total number of steps, and find that it's getting near to a British billion.

References

[1] E. R. Berlekamp, J. H. Conway and R. K. Guy, Winning Ways for your Mathematical Plays, Academic Press, London, 1982.

[2] J. H. Conway, Problem 2.4, The Math. Intelligencer, 3 (1980) 45.

[3] Martin Davis, Computability, Courant Inst. Math. Sci., N.Y. Univ., 1974, especially pp. 93–103.

[4] Marvin L. Minsky, Computation: Finite and Infinite Machines, Prentice-Hall, Englewood Cliffs, N. J., 1967.

[5] Sherman K. Stein, Mathematics: The Man-made Universe, W. H. Freeman, San Francisco, 1969, p. 42, 1976, p. 15.

[6] Klaus Wohlfahrt, The Math. Intelligencer, 3 (1981) 95.

Approximations to e

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Most undergraduates have been exposed to the monotonic nature of the approximations $\left(1+\frac{1}{n}\right)^n$ and $\left(1+\frac{1}{n}\right)^{n+1}$ of e, and certainly much literature has been written in this area, e.g., [2], [6]. Somewhat less well known, certainly at the undergraduate level, is the fact that

$$\left(1+\frac{1}{n}\right)^{n+\alpha} \textit{ decreases monotonically to } e \textit{ if and only if } \alpha \geqslant \frac{1}{2}.$$

This result appears as problem 168 in [8], and Dence [7] extensively investigates the monotonic nature of this approximation to e. We will use geometrically obvious properties of convex functions (whose proofs are well known and require only a standard first semester course in calculus) to establish this result.

It is easily shown that if the second derivative of a function f is nonnegative for $a \leqslant t \leqslant b$ (see Figure 1), then

$$(b-a)f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(t)\,dt \leqslant (b-a)\frac{f(a)+f(b)}{2}. \tag{1}$$

Geometrically, $(b-a)\,f\left(\frac{a+b}{2}\right)$ and $(b-a)\,\frac{f(a)+f(b)}{2}$ are, respectively, the areas of the interior and exterior trapezoidal approximations to the area under the curve $y=f(t)$ if f is nonnegative. Analytically, define

$$F(t)=\int_a^t f(x)-(t-a)f\left(\frac{a+t}{2}\right)dx, \qquad a \leqslant t \leqslant b.$$

Clearly, $F(a)=0$ and an application of the fundamental theorem of calculus together with the mean value theorem for derivatives quickly shows $F' \geqslant 0$; hence $F(b) \geqslant 0$. This proves the inequality on the left of (1); the inequality on the right follows from a similar argument. The inequalities in (1) are reversed if $f''(t) \leqslant 0$, $a \leqslant t \leqslant b$ (see [1], p. 179).

Now let $f_1(t)=1/t$, where $t>0$, and use (1) to obtain

$$(b-a)\frac{2}{a+b} < \int_a^b \frac{1}{t}\,dt < (b-a)\frac{a+b}{2ab}, \qquad a>0. \tag{2}$$

Applying the reversed inequalities of (1) to the function $f_2(t)=\ln t$, $t>0$ gives

$$(b-a)\frac{\ln a+\ln b}{2} < \int_a^b \ln t\,dt < (b-a)\ln\left(\frac{a+b}{2}\right), \qquad a>0. \tag{3}$$

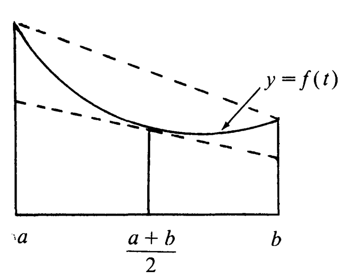

Figure 1

Corresponding to the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^{n+\alpha}, \qquad n = 1, 2, \ldots,$$

with α a real number we define $g(x) = \left(1 + \frac{1}{x}\right)^{x+\alpha}$ for $x > 0$. The sign of the derivative g' is determined by the sign of

$$\frac{d}{dx}\left[(x+\alpha)\ln\left(1 + \frac{1}{x}\right)\right] = \int_x^{x+\alpha} \frac{1}{t}\,dt + \int_{x+\alpha}^{x+1} \frac{1}{t}\,dt - \frac{(x+\alpha)}{x(x+1)}, \qquad 0 \leqslant \alpha \leqslant 1.$$

Applying (2), it follows that

$$\frac{-(x+\alpha)}{x(x+1)} + \alpha\left[\frac{2}{2x+\alpha}\right] + (1-\alpha)\left[\frac{2}{2x+\alpha+1}\right] < \frac{d}{dx}\left[(x+\alpha)\ln\left(1+\frac{1}{x}\right)\right]$$

$$< \frac{-(x+\alpha)}{x(x+1)} + \alpha\left[\frac{2x+\alpha}{2x(x+\alpha)}\right] + (1-\alpha)\left[\frac{2x+\alpha+1}{2(x+\alpha)(x+1)}\right] \tag{4}$$

where $0 \leqslant \alpha \leqslant 1$.

But the right-hand side of (4) is nonpositive whenever $\frac{1}{2} \leqslant \alpha \leqslant 1$ and the left-hand side of (4) is nonnegative for $0 \leqslant \alpha < \frac{1}{2}$ and x sufficiently large. Obviously when $\alpha < 0$, we have $g' > 0$ and when $\alpha > 1$, $g' < 0$. Consequently, $\{a_n\}$ decreases monotonically to e if and only if $\alpha \geqslant \frac{1}{2}$.

One nice application of the fact that $\left(1 + \frac{1}{n}\right)^{n+(1/2)}$ decreases monotonically to e is to show very easily that the sequence

$$b_n = \frac{n^{n+(1/2)}}{n!e^n}$$

is nondecreasing and bounded above:

$$\ln[eb_n] < \ln\left[\left(1+\frac{1}{n}\right)^{n+(1/2)} b_n\right] = \ln eb_{n+1} = \left(n+\tfrac{1}{2}\right)\int_n^{n+1}\frac{1}{t}\,dt + \ln b_n$$

$$< \left(n+\tfrac{1}{2}\right)\left(\frac{\frac{1}{n}+\frac{1}{n+1}}{2}\right) + \ln b_n = \ln\left[e^{1/4n(n+1)}b_n e\right].$$

So,

$$b_n < b_{n+1} < e^{1/4n(n+1)}b_n < \cdots < e^{1/4}b_1.$$

Coleman [**3**] and Courant and John [**4**] have very nice treatments of this. Now we can use Wallis' Formula ([**5**], pp. 408–410) to conclude that the sequence $\{b_n\}$ converges to $b = 1/\sqrt{2\pi}$.

More complicated sequences, such as

$$\left(1+\frac{\alpha}{n}\right)^{\beta n+\sigma}, \left(1+\frac{1}{n}\right)^{n+\frac{1}{n}}, \left(1+\frac{\alpha}{n}\right)^{n^\beta}, \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{c}{n}\right), \left(1+\frac{1}{n}\right)^{\ln n}, \left(1+\frac{1}{n}\right)^{\alpha(n)},$$

can be dealt with in a similar fashion (see also [**8**], problems 168, 169, and 172, page 38).

References

[1] R. C. Buck, Advanced Calculus, 3rd ed., McGraw-Hill, New York, 1978.

[2] J. D. Buckholtz, Geometric approximations for e, this MAGAZINE, 47 (1974), 138-143.

[3] A. J. Coleman, A simple proof of Stirling's formula, Amer. Math. Monthly, 58 (1951) 334-336. (Reprinted in Apostol, et al., Selected Papers on Calculus, MAA, 1969, pp. 325-327.)

[4] R. Courant and F. John, Introduction to Calculus and Analysis, vol. 1, Wiley, New York, 1965.

[5] A. Cruse and M. Granberg, Lectures on Freshman Calculus, Addison-Wesley, London, 1971.
[6] R. B. Darst, Comparison of estimates for e and e^{-1}, Amer. Math. Monthly, 86 (1979) 772-773.
[7] T. P. Dence, On the monotonicity of a class of exponential sequences, Amer. Math. Monthly, 88 (1981) 341-344.
[8] G. Pólya and G. Szegö, Problems and Theorems in Analysis, vol. 1, Wiley, New York, 1965.

Construction of Smith Numbers

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A. Wilansky defined a **Smith number** [3] to be a composite number, the sum of whose digits is equal to the sum of the digits of a prime factorization. The largest Smith number in print, the phone number of H. Smith, is

$$4937775 = 3 \times 5 \times 5 \times 65837.$$

The purpose of this note is to present techniques for the construction of Smith numbers. These techniques will yield a 362-digit Smith number.

For an integer n, let $S(n)$ denote the sum of the digits of n and $S_P(n)$ denote the sum of the digits of a prime factorization of n. Thus, a composite number n such that $S(n) = S_P(n)$ is a Smith number.

Smith numbers are easily generated from **prime repunits**. A repunit, denoted R_n, is a number all of whose digits are one. Several efforts have been made to discover repunits

$$R_n = \underbrace{111\ldots11}_{n \text{ ones}}$$

which are prime (see [2] and [5]). To date R_2, R_{19}, R_{23}, and R_{317} are known to be prime. H. C. Williams, who demonstrated the primality of R_{317}, is currently working on demonstrating the primality of R_{1031} [4].

Theorem 1. *If R_n is prime and $n \geqslant 3$, then $3304 \times R_n$ is a Smith number.*

Proof. For $n \geqslant 3$, it is easy to verify that $S(3304 \times R_n) = n + 27$ since $3304 = 2 \times 2 \times 2 \times 7 \times 59$, $S_P(3304 \times R_n) = n + 27$ for R_n prime.

It is an instructive exercise to show that *the multiplier* 3304 *may be replaced by a k-digit number M in Theorem* 1 *when $n \geqslant k-1$, if M has the following properties*:

1. *the sum of the digits is ten*, $(S(M) = 10)$;
2. *the sum of the digits of a prime factorization is* $9(k-1)$, $(S_P(M) = 9(k-1))$;
3. *the last digit is not zero.*

It would be interesting to know if there are infinitely many such multipliers (3304 is the smallest). However, the use of such multipliers for constructing Smith numbers appears to be limited by the requirement of prime repunits with many digits.

In the search for Smith numbers, some failures, such as 69, are better than others. Since $S(69) - S_P(69) = 7$, it follows that 690 is a Smith number. Any number which fails by a multiple of 7, i.e., $S(n) - S_P(n)$ is a positive multiple of 7, yields a Smith number when multiplied by an appropriate power of ten.

Theorem 2. *If $S(n) > S_P(n)$ and $S(n) \equiv S_P(n)$ mod 7, then $10^k \times n$ is a Smith number where $k = (S(n) - S_P(n))/7$.*

[5] A. Cruse and M. Granberg, Lectures on Freshman Calculus, Addison-Wesley, London, 1971.
[6] R. B. Darst, Comparison of estimates for e and e^{-1}, Amer. Math. Monthly, 86 (1979) 772-773.
[7] T. P. Dence, On the monotonicity of a class of exponential sequences, Amer. Math. Monthly, 88 (1981) 341-344.
[8] G. Pólya and G. Szegö, Problems and Theorems in Analysis, vol. 1, Wiley, New York, 1965.

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Smith numbers are easily generated from **prime repunits**. A repunit, denoted R_n, is a number all of whose digits are one. Several efforts have been made to discover repunits

$$R_n = \underbrace{111\dots11}_{n\text{ ones}}$$

which are prime (see [2] and [5]). To date R_2, R_{19}, R_{23}, and R_{317} are known to be prime. H. C. Williams, who demonstrated the primality of R_{317}, is currently working on demonstrating the primality of R_{1031} [4].

THEOREM 1. *If R_n is prime and $n \geqslant 3$, then $3304 \times R_n$ is a Smith number.*

Proof. For $n \geqslant 3$, it is easy to verify that $S(3304 \times R_n) = n + 27$ since $3304 = 2 \times 2 \times 2 \times 7 \times 59$, $S_P(3304 \times R_n) = n + 27$ for R_n prime.

It is an instructive exercise to show that *the multiplier* 3304 *may be replaced by a k-digit number M in Theorem* 1 *when $n \geqslant k - 1$, if M has the following properties*:

1. *the sum of the digits is ten*, $(S(M) = 10)$;
2. *the sum of the digits of a prime factorization is* $9(k-1)$, $(S_P(M) = 9(k-1))$;
3. *the last digit is not zero.*

It would be interesting to know if there are infinitely many such multipliers (3304 is the smallest). However, the use of such multipliers for constructing Smith numbers appears to be limited by the requirement of prime repunits with many digits.

In the search for Smith numbers, some failures, such as 69, are better than others. Since $S(69) - S_P(69) = 7$, it follows that 690 is a Smith number. Any number which fails by a multiple of 7, i.e., $S(n) - S_P(n)$ is a positive multiple of 7, yields a Smith number when multiplied by an appropriate power of ten.

THEOREM 2. *If $S(n) > S_P(n)$ and $S(n) \equiv S_P(n)$ mod 7, then $10^k \times n$ is a Smith number where $k = (S(n) - S_P(n))/7$.*

Proof. $S(10^k \times n) = S(n) = S_P(n) + 7k = S_P(10^k \times n)$.

Prime repunits are helpful in the construction of Smith numbers because the sum of the digits of certain multiples is predictable. Unfortunately, repunits which are prime are difficult to find. The collection of numbers consisting entirely of digits 0 and 1 retain some predictability in the sum of the digits of multiples and offer a larger collection of numbers in which to search for primes. Although such numbers are not amenable to a single multiplier production of Smith numbers, they can readily be converted to numbers which satisfy the hypothesis of Theorem 2.

THEOREM 3. *Let q be a prime all of whose digits are* 0 *or* 1. *There is a multiple of q which is a Smith number.*

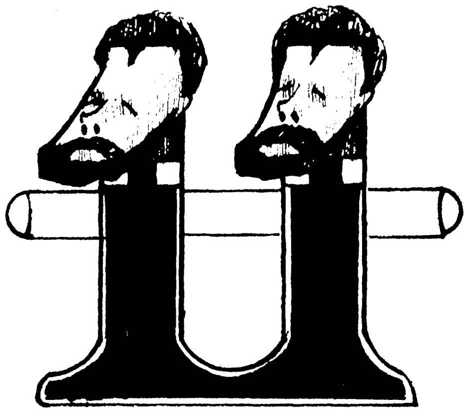
Proof. Let m be the number of digits of q which are 1, i.e., $S(q) = m$. By Theorem 2, it suffices to show that there exists a positive integer n such that $S(nq) \equiv S_P(nq)$ mod 7. If $m \equiv 0$ mod 7, then $S(7q) = 7m$ and $S_P(7q) = m + 7$. Since m cannot be 1, $S(7q) > S_P(7q)$ and $S(7q) \equiv S_P(7q)$ mod 7. Similarly, for m congruent to either 1, 2, 3, 4, 5, or 6 modulo 7, let $n = 6, 2, 5, 14, 3, 4$ respectively.

Since R_{317} is prime, Theorems 2 and 3 imply that $2 \times R_{317} \times 10^{45}$ is a Smith number.

The question of whether there are infinitely many Smith numbers or not remains open. A positive answer could be obtained from a sufficiently general construction. A positive answer could also be obtained by showing that there are infinitely many primes consisting of the two digits zero and one, an interesting and challenging problem in itself.

References

[1] Albert H. Beiler, Recreations in the Theory of Numbers, Dover, New York, 1964.
[2] Martin Gardner, Mathematical Games, Sci. Amer., 210 (March 1964) 127; 210 (June 1964) 118.
[3] A. Wilansky, Smith numbers, Two-Year Coll. Math. J., 13 (1982) 21.
[4] H. C. Williams, Some primes with interesting digit patterns, Math. Comp., 32 (1978) 1306-1310.
[5] Samuel Yates, The mystique of repunits, this MAGAZINE, 51 (1978) 22-28.
[6] ______, Periods of unique primes (letter), this MAGAZINE, 53 (1980) 314.



***Prime* observation 3. *Prime* ministers.**

An Intuitive Proof of Brouwer's Fixed Point Theorem in R^2

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Fixed point theorems play a major role in general equilibrium theory. Brouwer's theorem is the most basic of these; it states that any continuous function mapping a closed bounded convex set into itself must contain at least one fixed point (i.e., a point that is its own image).

Elementary discussions invariably give an intuitive proof of the theorem for functions of a single variable, as illustrated in FIGURE 1. In R^1 a set is convex if and only if it is an interval; thus a continuous mapping of the closed bounded interval $[x_0, x_1]$ into itself can be represented by a curve f. Since f connects the left-hand side of the rectangle to the right-hand side of the rectangle, it is intuitively obvious that f must intersect the diagonal of the rectangle at least once, and at this point $f(x^*) = x^*$. A bit more formally, if $f(x_0) \neq x_0$ and $f(x_1) \neq x_1$, then $\phi(x_0) = f(x_0) - x_0 > 0$ and $\phi(x_1) = f(x_1) - x_1 < 0$. Since ϕ is continuous on $[x_0, x_1]$, the intermediate value theorem implies that ϕ must assume the value zero somewhere on the open interval (x_0, x_1), which proves the theorem.

An intermediate- or advanced-level student should be a bit street-wise and skeptical of the validity of demonstrations based on two-dimensional diagrams. The purpose of this note is to demonstrate that the intuitive graphic proof generalizes to three dimensions (i.e., to functions on R^2) and can be made rigorous at that level.

To begin, let **W** be any closed bounded (i.e., compact) convex set in R^2 and let f be any continuous function mapping **W** into itself. Since **W** is bounded it can be contained in a rectangle as shown in FIGURE 2. We may now extend f to the closed rectangle $ABCD$ as follows. Choose an arbitrary interior point a in **W** and for each point b in the rectangle but not in **W**, define $f(b)$ to be the image of the point c at which the line through a and b intersects the boundary of **W**. The

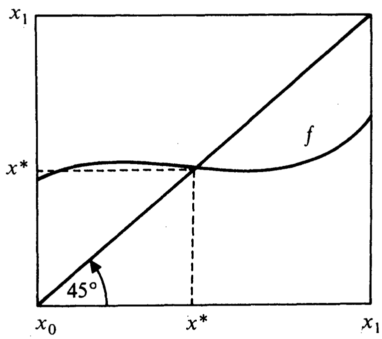

FIGURE 1

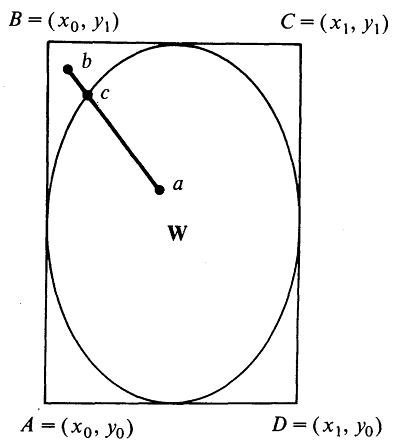

FIGURE 2

extended mapping is continuous and maps the closed rectangle $ABCD$ into $\mathbf{W}$. For simplicity we denote the extended mapping by f and represent f by $f(x, y) = (x', y')$ where

$$x' = g(x, y)$$
$$y' = h(x, y) \qquad (1)$$

with $x, x' \in [x_0, x_1]$; $y, y' \in [y_0, y_1]$; and with g and h continuous.

FIGURE 3 gives a three-dimensional representation with $\mathbf{W}$ and the rectangle $ABCD$ in the horizontal coordinate plane. Represented above the two-dimensional rectangle $ABCD$ is the three-dimensional box $EIJFGKLH$. The sides EI, FJ, GK, and HL as well as EH, IL, JK, and FG all correspond to the interval $[x_0, x_1]$. Similarly, the sides EF, IJ, LK, and HG all correspond to the interval $[y_0, y_1]$. The graph of g is given by the surface $MNOP$ which is restricted to the closed three-dimensional box since x' is restricted to $[x_0, x_1]$.

Now consider the projection mappings p_x, p_y defined by

$$x = p_x(x, y)$$
$$y = p_y(x, y). \qquad (2)$$

The graph of p_x in FIGURE 3 is the diagonal plane $EFKL$ and the intersection of g and p_x is the manifold RQ which projects into the horizontal coordinate plane as TS. Since neither the surface $MNOP$ nor the diagonal plane $EFKL$ have any rips in them, it is intuitively obvious that the intersection of g and p_x must connect the face and back of the three-dimensional box and that the projection TS connects opposite sides of $ABCD$. Further, TS represents the points (x, y) in $ABCD$ for which $x' = x$. Similarly, the intersection of h and p_y projected to the coordinate plane will connect the left and right sides of $ABCD$ as UV does in FIGURE 3. This projection represents the points (x, y) in $ABCD$ for which $y' = y$. Again, intuition tells us that UV must intersect TS (at least once) and any intersection of UV and TS is a fixed point of f.

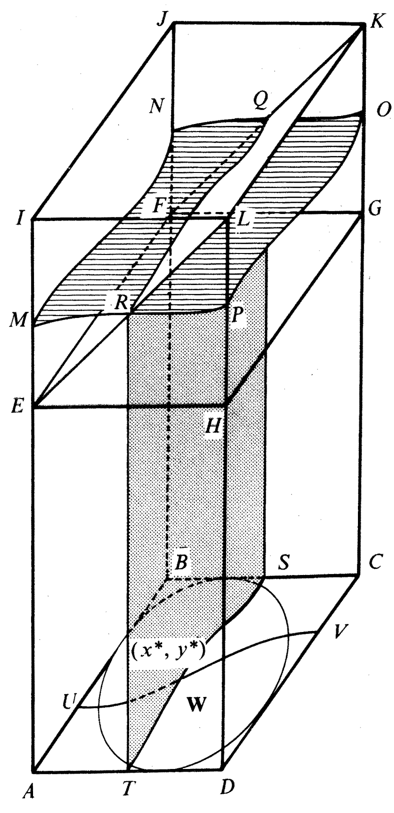

FIGURE 3

To make this demonstration rigorous, it is necessary to prove that TS (or UV) actually connects opposite sides of $ABCD$. As a first step we show that if Brouwer's theorem holds for functions which are "very close" to f, then it must hold for f itself. Let $\|q-r\|$ denote the usual Euclidean distance between two points q, r in R^2. For any given $\varepsilon>0$, suppose that there exists a continuous function $f^*: ABCD \to ABCD$ such that $\|f^*(x,y)-f(x,y)\| \leqslant \varepsilon$ for all $(x,y)\in ABCD$, and such that f^* has a fixed point in $ABCD$. We claim that this property implies that f has a fixed point in $ABCD$. Applying the property, we can assume that for each $n=1,2,3,\ldots$ there exists a continuous function $f_n: ABCD \to ABCD$ such that $\|f_n(x,y)-f(x,y)\| \leqslant 1/n$ for all $(x,y)\in ABCD$, and there is a point $Z_n \in ABCD$ such that $f_n(Z_n)=Z_n$. The compactness of $ABCD$ implies that the sequence $\{Z_n\}$ has a limit point, Z^*. We invite the reader to show that Z^* is a fixed point of f.

It is thus sufficient to replace f by another function which closely approximates f, then prove the Brouwer theorem for the replacement function. The Weierstrass approximation theorem (a generalized version is proven in [**4**, §36]; for the specific R^2 case see [**2**, p. 187, problem 2]) yields, for a given $\varepsilon>0$, a function $\hat{f}=(\hat{f}_1,\hat{f}_2)$: $ABCD \to R^2$ such that $\|f(x,y)-\hat{f}(x,y)\| \leqslant \varepsilon$ for all $(x,y)\in ABCD$, with $\hat{f}_1$ and $\hat{f}_2$ polynomials in x and y. However $\hat{f}$ may give values lying at a distance ε outside of $ABCD$, so we must shrink its range slightly. To do this, replace $\hat{f}_1(x,y)$ by

$$\frac{x_0+x_1}{2}+(x_1-x_0)\left(\frac{\hat{f}_1(x,y)-\dfrac{x_0+x_1}{2}}{x_1-x_0+4\varepsilon}\right)$$

and replace $\hat{f}_2$ by a similar expression. A short calculation shows that these new functions (which for simplicity we again call $\hat{f}_1$ and $\hat{f}_2$) approximate f and give us $\hat{f}=(\hat{f}_1,\hat{f}_2)$: $ABCD \to$ interior of $ABCD$.

Now define $\hat{g}$ on $ABCD$ by:

$$\hat{g}(x,y)=\hat{f}_1(x,y)-x.$$

Then

$$\hat{g}>0 \text{ on } AB \text{ and } \hat{g}<0 \text{ on } CD. \tag{3}$$

We must modify $\hat{g}$ still further so that its partial derivatives satisfy certain conditions, while retaining property (3). First, on AD, $y=y_0$ is constant so on AD $\hat{g}(x,y)=\hat{g}(x,y_0)$ is just a polynomial in x. By altering $\hat{g}(x,y)$ slightly if necessary we can ensure that $\hat{g}(x,y_0)$ has no repeated factors. There are then no points on AD where $\hat{g}(x,y)$ and $\partial\hat{g}(x,y)/\partial x$ vanish simultaneously. A further slight perturbation of $\hat{g}$ will ensure that at least one partial derivative of $\hat{g}$ is nonzero at each point in $ABCD$ where $\hat{g}(x,y)=0$. This assertion follows from Sard's theorem ([**3**], [**6**, Chapter 13, §14]; or for a proof of a special case of this theorem which can easily be adapted to the present situation, see [**1**, p. 35]).

We can now proceed directly. By the implicit function theorem (see any advanced calculus book) the above condition on $\partial\hat{g}/\partial x$ and $\partial\hat{g}/\partial y$ in $ABCD$ guarantees that $\hat{g}^{-1}(0)$ is a simple one-dimensional curve in a neighborhood of each point on $\hat{g}^{-1}(0)$. Consequently $\hat{g}^{-1}(0)$ is a collection of simple curves, no two of which intersect. Wherever one of these curves intersects AD, our earlier condition on $\partial\hat{g}/\partial x$ guarantees that the curve is not tangent to AD. By (3) and the $\partial\hat{g}/\partial x$ condition, the number of points in $AD \cap \hat{g}^{-1}(0)$ is odd, since at each such point $\hat{g}(x,y_0)$ changes sign. Curves which originate and terminate on AD account for an even number of these points so there must be a curve that has only one endpoint on AD. The other end of this curve cannot be on AB or CD by (3), so it must join AD and BC and we are free to label the endpoints T and S.

To complete the proof, we define $\hat{h}(x,y)=\hat{f}_2(x,y)-y$. Clearly $\hat{h}$ is continuous. Since $\hat{h}(S)\leqslant 0\leqslant \hat{h}(T)$, the intermediate value theorem assures us of the existence of at least one point (x^*,y^*) on TS such that $\hat{h}(x^*,y^*)=0$ or (equivalently) $\hat{f}_2(x^*,y^*)=y^*$. Since all points (x,y) on TS satisfy $\hat{g}(x,y)=x$, we have $\hat{g}(x^*,y^*)=x^*$. Thus there exists at least one point (x^*,y^*) in $ABCD$ such that $(x^*,y^*)=\hat{f}(x^*,y^*)$. Since $\hat{f}$: $ABCD \to \mathbf{W}$ we must have $(x^*,y^*)\in \mathbf{W}$. Thus

(x^*, y^*) is a fixed point of $\hat{f}$.

It is well known that in the class of compact sets, the fixed point property is not restricted only to convex sets [**5**, p. 9]. It can be shown that if a set has the fixed point property, then any set to which it is homeomorphic also has the fixed point property [**5**, p. 9]. This theorem can be used to prove that various plane sets with amoeboid shapes have the fixed point property. Our proof given above shows that *any bounded set S in R^2 having an interior point x such that each ray from x has only one intersecting point with the boundary of S has the fixed point property.*

References

[1] M. S. Berger and M. S. Berger, Perspectives in Nonlinearity, Benjamin, New York, 1968.
[2] E. Isaacson and H. B. Keller, Analysis of Numerical Methods, Wiley, New York, 1966.
[3] A. Sard, The measure of the critical points of differentiable maps, Bull. Amer. Math. Soc., 48 (1942) 883-890.
[4] G. F. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill, New York, 1963.
[5] D. R. Smart, Fixed Point Theorems, Cambridge Univ. Press, 1974.
[6] K. T. Smith, Smith's Primer of Modern Analysis, Bogden and Quigley, Tarrytown-on-Hudson, NY, 1971.

An Application of the Dominated Convergence Theorem to Mathematical Statistics

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The t-distribution arises in statistical inference problems involving random samples from a normal distribution. As is pointed out in nearly every elementary statistics course, the t-distribution converges (in the sense explained below) to the standard normal distribution as the number of degrees of freedom (roughly, the sample size) increases without bound. The main purpose of this note is to present a simple, conceptual proof of this fact—one that is accessible and appealing to undergraduates and does not require Stirling's approximation formula. The proof uses only the Dominated Convergence Theorem from analysis (see, for example, [**3**]). Indeed, a complementary purpose of this note is to exhibit a "real life" application of this theorem, a result deserving mention, if not proof, in an introductory real analysis course. Students unfamiliar with the theorem can follow our proof if they are willing to believe that limits commute with integrals under appropriate conditions.

A continuous probability distribution can be specified by giving its density function f. The probability that the random variable will assume a value between a and b is $\int_a^b f(x)\,dx$. In particular, f is nonnegative and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. The density function for the standard normal distribution is

$$\phi(x) = (2\pi)^{-1/2} e^{-x^2/2},$$

while the density function for the t-distribution with n degrees of freedom is

$$f_n(x) = C(n)\,g_n(x)$$

where

(x^*, y^*) is a fixed point of $\hat{f}$.

It is well known that in the class of compact sets, the fixed point property is not restricted only to convex sets [**5**, p. 9]. It can be shown that if a set has the fixed point property, then any set to which it is homeomorphic also has the fixed point property [**5**, p. 9]. This theorem can be used to prove that various plane sets with amoeboid shapes have the fixed point property. Our proof given above shows that *any bounded set S in R^2 having an interior point x such that each ray from x has only one intersecting point with the boundary of S has the fixed point property.*

References

[1] M. S. Berger and M. S. Berger, Perspectives in Nonlinearity, Benjamin, New York, 1968.
[2] E. Isaacson and H. B. Keller, Analysis of Numerical Methods, Wiley, New York, 1966.
[3] A. Sard, The measure of the critical points of differentiable maps, Bull. Amer. Math. Soc., 48 (1942) 883-890.
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A continuous probability distribution can be specified by giving its density function f. The probability that the random variable will assume a value between a and b is $\int_a^b f(x)\,dx$. In particular, f is nonnegative and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. The density function for the standard normal distribution is

$$\phi(x) = (2\pi)^{-1/2} e^{-x^2/2},$$

while the density function for the t-distribution with n degrees of freedom is

$$f_n(x) = C(n)\,g_n(x)$$

where

$$g_n(x) = (1 + x^2/n)^{-\frac{n+1}{2}}$$

$$C(n) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}$$

and Γ is the gamma function. We will show that $f_n(x) \to \phi(x)$ for all x as $n \to \infty$.

Most introductory statistics textbooks state this result and appeal to graphs and tables to suggest its plausibility. A few mathematical statistics texts provide a proof or assign one as an exercise (see [**1**] or [**2**], for example). It is easy enough to show that

$$\lim_{n\to\infty} g_n(x) = e^{-x^2/2};$$

one simply takes the natural logarithm and applies l'Hôpital's rule. The standard proof that $C(n) \to (2\pi)^{-1/2}$, however, relies on Stirling's formula (a nontrivial result) to approximate the gamma function. In fact, it is not necessary to know anything about $C(n)$. Assume for a moment that we can validly interchange limit and integral. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\,dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} g_n(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = (2\pi)^{1/2}.$$

But since $\int_{-\infty}^{\infty} C(n) g_n(x)\,dx = 1$, it follows that $\lim_{n\to\infty} C(n) = (2\pi)^{-1/2}$. The Dominated Convergence Theorem justifies interchanging limit and integral. We note that

$$g_n(x) = (1 + x^2/n)^{-\frac{n+1}{2}} < \left(1 + \frac{n+1}{n}x^2 + \frac{(n+1)n}{2n^2}x^4\right)^{-1/2}$$
$$< (1 + x^4/2)^{-1/2}$$

and this dominant function is integrable on $(-\infty, \infty)$.

Once it has been shown that $C(n) \to (2\pi)^{-1/2}$, it is an instructive (and nontrivial) exercise to use the recurrence relation for the gamma function,

$$\Gamma(u+1) = u\Gamma(u)$$

to show that $C(n)$ is monotonically increasing. (Hint: First derive the easy recurrence relations

$$C(n) = \frac{1}{2\pi}\left(\frac{n-1}{n}\right)^{1/2} \frac{1}{C(n-1)} = \left[\frac{(n-1)^2}{n(n-2)}\right]^{1/2} C(n-2).$$

Then show that for $u > v$ the sequence $\{C(u+2k)/C(v+2k)\}$ is monotonically decreasing to 1, and hence that $C(u)/C(v) > 1$.) As a corollary it is easy to prove that $C(n)$ obeys the inequalities

$$(2\pi)^{-1/2}\left[\frac{n-1}{n}\right]^{1/4} < C(n) < (2\pi)^{-1/2}\left[\frac{n}{n+1}\right]^{1/4}.$$

These algebraic bounds for $C(n)$ in fact provide a much better estimate for $C(n)$ than that given by the most elementary version of Stirling's formula

$$\Gamma(u) \approx (2\pi)^{-1/2} u^{u-1/2} e^{-u}.$$

In fact, the arithmetic average of the two bounds has an error of less than 10^{-4} for $n \geqslant 8$.

References

[1] Harald Cramér, Mathematical Methods of Statistics, Princeton University Press, 1951, pp. 251–252.

[2] B. W. Lindgren, Statistical Theory, 2nd ed., Macmillan, 1968, pp. 381–382.

[3] Robert M. McLeod, The Generalized Riemann Integral, The Carus Mathematical Monographs, No. 23, Math. Assoc. of Amer., 1980, pp. 88–102.

A Nonconverging Newton Sequence

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Newton's method is one of the keystones of numerical solutions of equations. Provided that f is differentiable twice, we can assert that the sequence $\{x_n\}$, defined by

$$x_{n+1} = x_n - f(x_n)/f'(x_n), \tag{1}$$

converges to a solution of the equation $f(x) = 0$ if the initial term x_0 has been chosen happily enough. (See e.g., [**4**, p. 56].)

However, the behavior of the sequence can be interesting and worth studying even when there are no real solutions of the equation $f(x) = 0$. Quadratic equations give a nice example. The case of real roots is straightforward and nicely covered in [**3**, pp. 65–66]. On the other hand, we can have a quadratic equation

$$x^2 + px + q = 0 \tag{2}$$

with real coefficients and complex roots. It is enough, however, to study the equation

$$x^2 + 1 = 0, \tag{3}$$

since equation (2) is transformed to the form (3) by a change of variable, $y = (2x + p)/\sqrt{4q - p^2}$.

If we attempt to solve the equation $x^2 + 1 = 0$ using Newton's method, formula (1) takes the form

$$x_{n+1} = \left(x_n - x_n^{-1}\right)/2. \tag{4}$$

We start with a real initial term x_0, though of course there is no chance for convergence of the real sequence $\{x_n\}$ to the complex solutions $\pm i$.

Three different types of behavior of the sequence defined by (4) are possible:

(i) $x_n = 0$ for some n and hence the sequence terminates;
(ii) the sequence is periodic: $x_{n+p} = x_n$;
(iii) the sequence is nonterminating and aperiodic.

We shall investigate how the behavior of the sequence depends on the choice of the initial term x_0.

When the recurrence formula (4) is written in the form $x_{n+1} = (x_n^2 - 1)/2x_n$, it resembles the formula for the cotangent of a double angle:

$$\operatorname{ctg} 2t = \left(\operatorname{ctg}^2 t - 1\right)/2 \operatorname{ctg} t. \tag{5}$$

This observation permits us to introduce a new sequence $\{r_n\}$ defined as follows:

$$r_0 = \frac{1}{\pi} \operatorname{arc\,ctg} x_0$$

$$r_{n+1} = (2r_n) \text{ (fractional part)}. \tag{6}$$

The sequence (6) satisfies $0 < r_n < 1$ and it is related to the sequence (4) by the equation

$$x_n = \operatorname{ctg} \pi r_n. \tag{7}$$

Now consider the binary expansion of r_0:

$$r_0 = 0.a_1a_2a_3\ldots, \tag{8}$$

which is an infinite string of zeros and ones. Definition (6) shows that the term r_n is represented by the same string with the first n bits on the left cut off. This last observation permits us to classify the type of behavior of (4) very conveniently.

If the binary expansion (8) terminates, i.e., r_0 is a reduced fraction of denominator 2^k, then $r_k = 0$ and the sequence (4) has terminated one step before. The second case is when r_0 is a reduced fraction with denominator $L \cdot 2^k$, where $L > 1$ is an odd integer. The binary expansion (8) is periodic from a_k (inclusively) on and so is the sequence (4). Clearly, aperiodic behavior of (4) occurs exactly in the case of aperiodic expansion in (8), which in turn means that r_0 is irrational.

Thus we can sum up the classification as follows.

THEOREM 1. *The behavior of the sequence* (4) *follows from the binary expansion of* r_0:

(i) *the sequence terminates iff the expansion terminates*;
(ii) *the sequence is periodic iff the expansion is periodic* (*from some bit on*);
(iii) *the sequence is aperiodic iff the expansion is aperiodic, i.e.,* r_0 *is irrational.*

Note that the dependence of the behavior of (4) on x_0 is highly noncontinuous, since in any neighborhood of any real number we have binary expansions of all three types.

In the periodic case, one can ask two questions. What is the length of the period? How many periods of a given length exist?

Suppose, as above, $r_0 = S/(L \cdot 2^k)$ is in the reduced form with $L > 1$ an odd integer. The length of the period is equal to the smallest m for which $2^m \equiv 1 \bmod L$. To answer the other question, we construct all the periods of a given length p and discard those that are repetitions of shorter periods. Let the prime factorization of p be

$$p = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}, \alpha_i > 0$$

and let us adopt the notation $p_{ij\cdots m} = p/(q_i q_j \cdots q_m)$. Then the number of different periods of length p is given by the formula

$$N(p) = 2^p - \sum_i 2^{p_i} + \sum_{i,j} 2^{p_{ij}} - \sum_{i,j,k} 2^{p_{ijk}} + \cdots . \tag{9}$$

To see this, one can use either the Inclusion-Exclusion principle or the Möbius inversion formula (see [**1**, p. 236]). In the first case, we note that there are 2^p possible permutations of zeros and ones in the p-digit formula. If a period is a repetition of shorter periods, it can be represented as a repetition of a period of length p_i for some i. So we discard for each i those 2^{p_i} periods. And since in this way we discarded the periods of lengths p_{ij} twice, they have to be added once again, and so on.

To use the Möbius formula, we note that

$$\sum_{d|p} N(d) = 2^p,$$

the sum taken over all the divisors of p. Then applying the inversion formula, we get (9) once again in the form

$$N(p) = \sum_{d|p} \mu(p/d) 2^d,$$

where the Möbius function $\mu(d)$ is defined as: $\mu(d) = 0$ if d is not the product of distinct primes and $\mu(d) = (-1)^m$ if d is the product of m different primes.

It is interesting to compare formula (9) with [**2**], where the same expression is used to count the prime elements of a certain degree in a ring. In the same paper it is mentioned that the fact that $N(p)$ is divisible by p is a consequence of Gauss's generalization of the "little" Fermat theorem. In our case this is obvious, since $N(p)/p$ counts the number of cycles of length p.

The aperiodic case also gives rise to an interesting question. At first glance it might seem that the sequence (4) would be dense in the reals. Yet the following example shows that it is not so; in fact, our sequence $\{x_n\}$ below is even bounded.

EXAMPLE. We construct the irrational number

$$r_0 = 0.a_1 a_2 a_3 \ldots$$

by the rule: $a_{2k-1} = 0$, $a_{2k} = 1$ if k is not a square, and $a_{2k} = 0$ if k is a square. Thus

$$r_0 = 0.00010100010101010001 01\ldots.$$

By the definition of the sequence (6) we conclude that

$$r_n < 0.101010\ldots = 2/3.$$

Also, since there are never more than three consecutive zeros in the expansion of r_0, we conclude that

$$r_n > 0.0001 = 1/16.$$

The associated sequence $\{x_n\}$ defined by (7) for this example is bounded between $\operatorname{ctg} 2\pi/3$ and $\operatorname{ctg} \pi/16$.

Actually, the "first glance" was not so far off as the above example might suggest. We now investigate under what circumstances the sequence $\{x_n\}$ is dense in the reals; we shall find out that this happens fairly often.

Suppose that we have a real number b, lying between 0 and 1, with binary expansion

$$b = 0.b_1 b_2 b_3 \ldots.$$

We want b to be a limit point of the sequence $\{r_n\}$ (or equivalently, $\operatorname{ctg} \pi b$ to be a limit point of $\{x_n\}$). If we take $\epsilon = 2^{-k}$, the term r_n lies in the ϵ-neighborhood of b if the initial string $b_1 b_2 b_3 \ldots b_k$ is also the initial string of r_n. Since this has to be true for infinitely many terms r_n, it follows that the string $b_1 b_2 b_3 \ldots b_k$ has to appear infinitely many times in the binary expansion of r_0. Since this must happen for every k, any finite string from the beginning of the binary expansion of b has to appear infinitely often in the binary expansion of r_0. Thus we have

PROPOSITION 1. *A real number b with binary expansion* $b = 0.b_1 b_2 b_3 \ldots$ *is a limit point for the sequence* $\{r_n\}$ *defined by* (6) *iff every finite string* $b_1 b_2 b_3 \ldots b_k$ *appears infinitely many times in the binary expansion of* r_0.

To assure the density of $\{x_n\}$ in the reals, the criterion of Proposition 1 has to be fulfilled for every $b \in (0, 1)$, which can be stated as follows.

PROPOSITION 2. *The sequence* $\{x_n\}$ *defined in* (7) *is dense in the reals iff the binary expansion of* r_0 *contains every finite string infinitely many times.*

In [**1**, p. 124] a real number is defined to be **2-normal** if its binary expansion contains every string of length k infinitely many times and with the right relative frequency 2^{-k}. Further, it is proved there that *almost all real numbers are* 2-*normal*, i.e., the complement of the set of 2-normal numbers can be covered with a system of intervals of arbitrarily small total length. But 2-normal numbers *a fortiori* comply with the condition in Proposition 2, hence we can state our final result.

THEOREM 2. *For almost all* r_0 (*and therefore for almost all values of the initial term* x_0), *the sequence* $\{x_n\}$ *defined in* (7) (*or* (4)) *is dense in the set of real numbers.*

I wish to thank Professor Z. Bohte and the reviewer for many valuable suggestions.

References

[1] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, Clarendon Press, 1960.
[2] R. Isaacs, The distribution of primes in a special ring of integers, this MAGAZINE, 52 (1979) 31–36.
[3] Z. A. Melzak, Companion to Concrete Mathematics, John Wiley and Sons, New York, 1973.
[4] A. M. Ostrowski, Solutions of Equations in Euclidean and Banach Spaces, Academic Press, New York, 1973.

PROBLEMS

LEROY F. MEYERS, Editor
G. A. EDGAR, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before June 1, 1983.

1161. Two equilateral triangles are placed so that their intersection is a hexagon (not necessarily regular). The vertices of the equilateral triangles are connected to form an outer hexagon. Show that if three alternate angles of the outer hexagon are equal, then the triangles have the same center. [*Roger Izard, Dallas, Texas.*]

1162. Suppose R is a finite associative ring in which there is exactly one nonzero element which is neither a left nor a right divisor of zero. Show that R is Boolean, i.e., that $x^2 = x$ for all $x \in R$. [*Enzo R. Gentile, Buenos Aires, Argentina.*]

1163*. Prove or disprove: If f is a one-to-one function on R into R and f is continuous at some $b \in R$, then f^{-1}, the inverse of f, is continuous at $f(b)$. [*Michael Grossman, University of Lowell.*]

1164. Are there any numerals of two or more "digits" which represent primes in all sufficiently large bases? (Obviously, each prime is represented by a single "digit" in every base larger than the prime.) [*A. Joseph Berlau, Hartsdale, New York.*]

ASSISTANT EDITORS: DANIEL B. SHAPIRO and WILLIAM A. MCWORTER, JR., *The Ohio State University.*

We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) will be placed next to a problem number to indicate that the proposer did not supply a solution.

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address. It is not necessary to submit duplicate copies.

Send all communications to the problems department to Leroy F. Meyers, Mathematics Department, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q680. If a, b, $c > 0$, show that the system

$$ax + by + cxy = a + b + c$$
$$by + cz + ayz = a + b + c$$
$$cz + ax + bzx = a + b + c$$

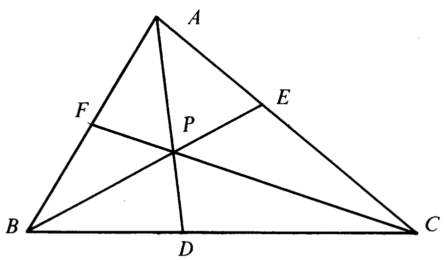
has the unique nonnegative solution $(x, y, z) = (1, 1, 1)$. [*Gregg Patruno, student, Princeton University.*]

Q681. Define a one-to-one function f from N onto $N \setminus \{1983\}$ so that $f(n)$ is expressed explicitly by a formula containing constants, n, some of the four arithmetic operations, and a single use of a common mathematical but extra-arithmetic operation. [*Robert Cabane, Paris, France.*]

Solutions

Proportions and Angle Bisectors **November 1981**

1132. Let triangle ABC be given and labeled as shown in the figure.



(a) Show that if AD bisects angle A and $BD \cdot CE = DC \cdot BF$, then ABC is an isosceles triangle.
(b) Show that if AD, BE, and CF bisect angles A, B, and C respectively, and $BP \cdot FP = BF \cdot AP$, then ABC is a right triangle. [*Roger Izard, Dallas, Texas.*]

Solution: (a) From the hypotheses and properties of proportions we have

$$\frac{AB}{AC} = \frac{BD}{DC} = \frac{BF}{CE} = \frac{AB - BF}{AC - CE} = \frac{AF}{AE}. \qquad (1)$$

Hence FE is parallel to BC, and so the triangles FPE and BPC are similar. Since AP bisects angles BAE and CAF, we get

$$\frac{AB}{AE} = \frac{BP}{PE} = \frac{CP}{PF} = \frac{AC}{AF},$$

and it follows that

$$\frac{AB}{AC} = \frac{AE}{AF}. \qquad (2)$$

From (1) and (2) it then follows that $AB = AC$.

M. S. K. SASTRY
Ohio University

Solution: (b) Clearly,

$$\angle FPB = \angle CBP + \angle BCP = \frac{1}{2}\angle B + \frac{1}{2}\angle C = \frac{\pi}{2} - \frac{1}{2}\angle A = \frac{\pi}{2} - \angle BAP.$$

Then from the hypothesis $BP{:}BF = AP{:}FP$ and the fact that angles FPB and BAP are acute, it follows that triangles ABP and PBF are similar. Hence

$$\angle BAP = \angle FPB = \frac{\pi}{2} - \angle BAP,$$

so that

$$\angle A = 2\angle BAP = \frac{\pi}{2}.$$

L. KUIPERS
Sierre, Switzerland

Also solved by F. S. Cater ((a) only), Howard Eves, Chico Problem Group, Robert Heller, Victor Hernandez (Spain), John P. Hoyt, W. C. Igips ((a) only), Mark Kantrowitz (student, (a) only), Hans Kappus (Switzerland), M. K. King (Kenya), Hiroshi Kotera (Japan), L. Kuipers (Switzerland, (a) as well), Kee-Wai Lau (Hong Kong), Henry S. Lieberman, Hubert J. Ludwig, Vania D. Mascioni (student, Switzerland, (b) only), Stanley Rabinowitz, M. S. K. Sastry ((b) as well), Robert S. Stacy (West Germany), J. M. Stark, Kao Hwa Sze, Dimitris Tsakmakis (Greece), Michael Vowe (Switzerland), and the proposer.

Most solvers of part (a) combined the hypothesis with Ceva's theorem to deduce $AF = AE$, and then used the proportion (1) in the solution above. Most solvers of part (b) used the law of sines on triangles ABP and PBF, equivalent to the side-side-angle similarity in the solution above. There were three incorrect solutions of part (a) and two of part (b).

Polynomial Properties — November 1981

1133. Let f_{nk}, $k = 1,\dots,n$, be n polynomials of degree n defined by

$$f_{nk}(x) = \frac{(-x)^k}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\left[\frac{(1-x)^n - 1}{x}\right].$$

Show the following properties of f_{nk}:

(a) For each x in $(0,1)$, $f_{n1}(x) > f_{n2}(x) > \cdots > f_{nn}(x)$.
(b) Each f_{nk} is strictly increasing on $[0,1]$ with $f_{nk}(0) = 0$ and $f_{nk}(1) = 1$.
(c) $\frac{1}{n}\sum_{k=1}^{n} f_{nk}(x) = x$.
(d) $\int_0^1 f_{nk}(x)\,dx = 1 - \frac{k}{n+1}$. [*G. Z. Chang, University of Utah.*]

Solution I: By directly carrying out the indicated differentiations in the definition of the proposer's sequence of nth degree polynomials, one obtains the explicit formula

$$f_{nk}(x) = \sum_{r=k}^{n} \binom{n}{r} x^r (1-x)^{n-r}, \quad (k = 1,2,\dots,n), \tag{1}$$

which can also be expressed (in terms of an integral representing the so-called incomplete Beta function) by

$$f_{nk}(x) = k\binom{n}{k}\int_0^x t^{k-1}(1-t)^{n-k}\,dt. \tag{2}$$

Equation (1) clearly gives $f_{nk}(0)=0$ and $f_{nk}(1)=1$. It is also seen from (1) that $f_{n1}(x)=1-(1-x)^n<1$ for each x in $(0,1)$, and that

$$f_{n,p}(x)-f_{n,p+1}(x)=\binom{n}{p}x^p(1-x)^{n-p}>0$$

when $p=1,2,\dots,n-1$. For $0<x<1$ we therefore have

$$1>f_{n1}(x)>f_{n2}(x)>\cdots>f_{nn}(x)>0.$$

Now equation (2) yields simply

$$\frac{d}{dx}f_{nk}(x)=k\binom{n}{k}x^{k-1}(1-x)^{n-k},$$

which is obviously positive for every x in $(0,1)$, thus showing that the polynomials f_{nk} are strictly increasing in $[0,1]$. Furthermore, using (2), one gets

$$\frac{1}{n}\sum_{k=1}^{n}f_{nk}(x)=\frac{1}{n}\int_0^x\frac{1}{t}\left[\sum_{k=1}^{n}k\binom{n}{k}t^k(1-t)^{n-k}\right]dt=\frac{1}{n}\int_0^x\frac{1}{t}(nt)\,dt=x.$$

Finally, by again employing (2), we have

$$\int_0^1 f_{nk}(x)\,dx=k\binom{n}{k}\int_0^1\int_0^x t^{k-1}(1-t)^{n-k}dt\,dx=k\binom{n}{k}\int_0^1\int_t^1 t^{k-1}(1-t)^{n-k}dx\,dt$$

$$=k\binom{n}{k}\int_0^1 t^{k-1}(1-t)^{n-k+1}dt=\frac{n-k+1}{n+1}=1-\frac{k}{n+1}.$$

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Solution II: We prove the statements in the order (c), (b), (a), (d).

For any polynomial $p(x)$ of degree $n-1$,

$$p(x+h)=\sum_{k=1}^{n}\frac{p^{(k-1)}(x)h^{k-1}}{(k-1)!}. \tag{1}$$

Applying (1) to $p(x)=[(1-x)^n-1]/x$ with $h=-tx$ we obtain the identity

$$\sum_{k=1}^{n}f_{nk}(x)t^{k-1}=\frac{1-[1-(1-t)x]^n}{1-t}. \tag{2}$$

Since the right-hand side of (2) reduces to nx for $t=1$, (c) is proved.

Now we differentiate (2) with respect to x:

$$\sum_{k=1}^{n}f'_{nk}(x)t^{k-1}=n[1-(1-t)x]^{n-1}. \tag{3}$$

This identity may be differentiated repeatedly with respect to t. Setting $t=0$ in the resulting equations directly leads to

$$f'_{nk}(x)=k\binom{n}{k}x^{k-1}(1-x)^{n-k}. \tag{4}$$

Hence

$$f'_{nk}(x)>0 \text{ for } x\in(0,1). \tag{5}$$

Since (2) is a polynomial identity in t we find, taking $x=0$ and $x=1$, respectively, the values $f_{nk}(0)=0$, $f_{nk}(1)=1$. Thus (b) is proved.

Next, from the definition of the f_{nk} we have the recursion

$$f_{n,k+1}(x)=f_{nk}(x)-(x/k)f'_{nk}(x). \tag{6}$$

Proposition (a) now follows from (5) and (6).

Finally let $\int_0^1 f_{nk}(x)\,dx = I_k$. Then $I_1 = 1 - 1/(n+1)$, by direct computation. Integrating (6) over [0, 1] (second term by parts), we obtain the recursion

$$I_{k+1} = (1 + 1/k) I_k - 1/k, \tag{7}$$

and proposition (d) is now easily verified by induction.

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Rodersdorf, Switzerland

Also solved by F. Lee Cook, Milton P. Eisner, Ralph Garfield, Chico Problem Group, Victor Hernandez (Spain), Hiroshi Kotera (Japan), L. Kuipers (Switzerland), Vania D. Mascioni (Switzerland, student), Zane C. Motteler, Roger B. Nelsen, Daniel A. Rawsthorne, Heinz-Jürgen Seiffert (West Germany, student), J. M. Stark, Kao Hwa Sze, Michael Vowe (Switzerland), and the proposer.

Hernandez interpreted his formula $\sum_{r=k-1}^{n-1} \binom{r}{k-1} x^k (1-x)^{r-k+1}$ for $f_{nk}(x)$ as the probability that the kth head occurs in the first n tosses of a coin in which x is the probability of tossing a head, where $x \in (0,1)$.

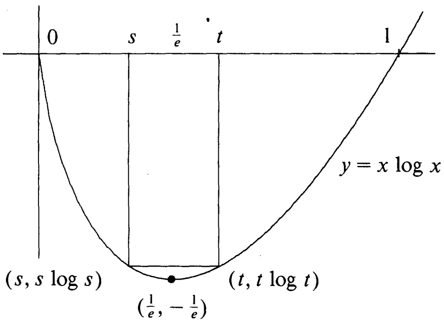
A Tricky Supremum — November 1981

1134. Let $a \geqslant 0$ be fixed. Find

$$\sup\left\{ \frac{\log s}{t^a} : 0 < s < t < 1 \text{ and } s \log s = t \log t \right\}.$$

[*Benjamin G. Klein, Davidson College.*]

Partial solution: We first treat the case in which $a = 1$. The graph of $y = x \cdot \log x$ is shown. By elementary calculus, there is a unique absolute minimum at the point $(1/e, -1/e)$. Then, with s and t as in the problem statement, we have $0 < s < 1/e < t < 1$.



We will consider s to depend on t in such a way that $s \cdot \log s = t \cdot \log t$ for $1/e < t < 1$. Then

$$\frac{ds}{dt} = \frac{1 + \log t}{1 + \log s} \text{ for } 1/e < t < 1.$$

We also set

$$F(t) = \frac{\log s}{t} = \frac{\log t}{s} \text{ for } 1/e < t < 1.$$

Then F has a limit of $-e$ from the right at $1/e$ and a limit of $-\infty$ from the left at 1, and

$$F'(t) = \frac{\dfrac{t}{s} \cdot \dfrac{ds}{dt} - \log s}{t^2} = \frac{t + t \cdot \log t - s \cdot \log s - s(\log s)^2}{st^2(1 + \log s)}$$

$$= \frac{1 - (\log s)(\log t)}{st(1 + \log s)}.$$

Suppose that F assumes a value greater than $-e$ in $(1/e, 1)$. Then there is t_0 in $(1/e, 1)$ such that $F(t_0) > -e$ and $F'(t_0) = 0$. With s_0 chosen in $(0, 1/e)$ so that

$$s_0 \log s_0 = t_0 \log t_0,$$

it follows from $F'(t_0) = 0$ that $1 = (\log s_0)(\log t_0)$, so that

$$s_0 = t_0(\log t_0)/(\log s_0) = t_0(\log t_0)^2,$$

whence

$$F(t_0) = \frac{\log t_0}{s_0} = \frac{1}{t_0 \log t_0} < -e < F(t_0),$$

a contradiction. Thus, $F(t) \leqslant -e$ for all t in $(1/e, 1)$, and, accordingly, $\sup F = -e$.

From the above argument we have $(\log s)/t \leqslant -e$ whenever $0 < s < 1/e < t < 1$ (with $s \log s = t \log t$), so that $et > 1$ and $-\log s \geqslant et \geqslant (et)^a$ for $a \leqslant 1$. Then $(\log s)/t^a \leqslant -e^a$. But

$$\lim_{t \downarrow 1/e} \frac{\log s}{t^a} = -e^a.$$

Hence $\sup F = -e^a$ if $a \leqslant 1$. (Note that a may be negative.)

The case in which $a > 1$ seems to be considerably more complicated. For example, with $a = 2$, the supremum is *not* achieved with $s = 1/e = t$. (Consider $t = 1/2$, so that $s = 1/4$. Then

$$(\log s)/t^2 = -4\log 4 > -5.6 > -7.3 > -e^2.)$$

BENJAMIN G. KLEIN
Davidson College

Also solved by the proposer (second partial solution). There were four incorrect or seriously incomplete solutions, mainly due to incorrect multiplication of an inequality by the negative number $\log s$.

Analytic Continuation of a Power Series **November 1981**

1135. Let $f(z) = \sum_{n=1}^{\infty}(1 - ab)(1 - a^2b) \cdots (1 - a^nb)z^n$, where $|a| < 1$ and $a^nb \neq 1$ for $n \geqslant 1$.

(a) Find the radius of convergence and show that f can be continued analytically to a meromorphic function f^* on $\mathbb{C}$, the complex plane.

(b) Find the residues at the poles of f^* and exhibit a series representation for f^*. [*Paul Schweitzer, University of Rochester.*]

Solution: If $ab = 0$, then the continuation of f is given by $f^*(z) = z/(1 - z)$, which has a simple pole at 1 with residue -1.

We now assume that $ab \neq 0$. The conditions on a and b insure that

$$c_\infty \equiv \lim_{n \to \infty} c_n, \text{ where } c_n \equiv \prod_{k=1}^{n} (1 - a^kb),$$

exists and is nonzero (since $\sum_{n=1}^{\infty} a^nb$ converges absolutely).

The ratio test shows that the series for $f(z)$ has radius of convergence unity. For $|z| < 1$, use $c_n = (1 - a^nb)c_{n-1}$ for $n \geqslant 2$ to obtain

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} c_n z^n = c_1 z + \sum_{n=2}^{\infty} (1 - a^nb)c_{n-1}z^n \\ &= (1 - ab)z + zf(z) - abzf(az). \end{aligned}$$

This result can also be obtained by summing the series by parts. Rewrite it as:

$$c_1 z + \sum_{n=2}^{\infty} (c_n - c_{n-1}) z^n = (1-z) f(z) = (1-ab)z - abzf(az). \tag{1}$$

The rightmost term is well-behaved for $|z| < 1/a$, leading to the conclusion that the leftmost sum converges when $z = 1$; hence

$$0 \neq c_\infty = \lim_{z \to 1} (1-z) f(z) = 1 - ab - abf(a).$$

Hence f has a simple pole at 1 with residue $-c_\infty$.

Rewrite (1) as

$$f(z) = \frac{(1-ab)z - abzf(az)}{1-z}. \tag{2}$$

Since the numerator is finite for $|z| < 1/a$, we can use (2) to extend f to a function f^* defined within the circle $|z| < 1/a$, except for a simple pole at 1.

Iteration of (2) produces

$$f(z) = (1-ab)\left[\frac{z}{1-z} - \frac{abz(az)}{(1-z)(1-az)} + \cdots + \frac{(-ab)^n z(az)\cdots(a^n z)}{(1-z)(1-az)\cdots(1-a^n z)}\right]$$
$$+ \frac{(-ab)^{n+1} z(az)\cdots(a^n z)}{(1-z)(1-az)\cdots(1-a^n z)} f(a^{n+1}z) \qquad \text{if } |z| < 1. \tag{3}$$

The ratio test demonstrates absolute convergence of the bracketed sum as $n \to \infty$, while the last term approaches $0f(0) = 0$. Thus

$$f(z) = (1-ab) \sum_{n=0}^{\infty} \frac{(-ab)^n z(az)\cdots(a^n z)}{(1-z)(1-az)\cdots(1-a^n z)} \tag{4}$$

represents $f(z)$ for all $|z| < 1$.

The sum in (4) converges absolutely by the ratio test, provided $z \notin \{a^{-k}\}_{k=0}^{\infty}$, and represents the desired analytic continuation f^* of f for all $z \notin \{a^{-k}\}_{k=0}^{\infty}$.

To show that f^* has simple poles at a^{-k} $(k = 1,2,3,\dots)$, we use (3) with $f^*(z)$ as left-hand side to write

$$\lim_{z \to a^{-k}} (1 - a^k z) f^*(z) = \left[\frac{(-ab)^k z(az)\cdots(a^k z)}{(1-z)(1-az)\cdots(1-a^{k-1}z)}\right]_{z=a^{-k}} (1 - ab - abf(a))$$
$$= \frac{(ab)^k c_\infty}{(1-a)(1-a^2)\cdots(1-a^k)} \neq 0, \quad k = 1,2,3,\dots.$$

Therefore the residue at a^{-k} is

$$\frac{-b^k c_\infty}{(1-a)(1-a^2)\cdots(1-a^k)}.$$

Finally, f^* has an essential singularity at ∞ (unless $ab = 0$) because every neighborhood of ∞ contains an infinite number of poles (at the points a^{-k}).

Paul Schweitzer
Graduate School of Management
University of Rochester

Also solved by J. M. Stark. One incorrect solution was received.

A Known Isomorphism January 1982

1136. Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers and let S^1 be the subgroup of $\mathbb{C}^*$ where the elements of S^1 have modulus one. Are $\mathbb{C}^*$ and S^1 isomorphic as abstract groups? [*William Horten and Daniel B. Shapiro, The Ohio State University.*]

Solution: The answer is yes.

Let R be the additive group of real numbers, Q the additive group of rational numbers, and Z the additive group of integers. Now R can be regarded as a vector space over Q, and R is the vector direct sum of c subspaces of dimension 1, one of which is Q. (The cardinal number of the continuum is denoted by c.) Then R is the group direct sum $Q \oplus \sum_a \oplus G_a$ where there are c groups G_a and each G_a is isomorphic to Q.

Let R^+ be the multiplicative group of positive real numbers. Then R^+ is isomorphic to R via $x \mapsto \log x$. Also S^1 is isomorphic to R/Z and hence is isomorphic to $(Q/Z) \oplus \sum_a \oplus G_a$. And $\mathbb{C}^*$ is isomorphic to $R^+ \oplus S^1$ because each nonzero complex number z is uniquely expressed as $z = |z|(\cos\theta + i\sin\theta)$ where $0 \leqslant \theta < 2\pi$, $|z| \in R^+$, and $\cos\theta + i\sin\theta \in S^1$.

Finally, $\mathbb{C}^*$ is isomorphic to $R^+ \oplus S^1$ and so to

$$\left(Q \oplus \sum_a \oplus G_a\right) \oplus \left((Q/Z) \oplus \sum_a \oplus G_a\right).$$

Since there are c groups G_a, $\mathbb{C}^*$ is isomorphic to

$$(Q/Z) \oplus \sum_a \oplus G_a$$

and to S^1.

Both $\mathbb{C}^*$ and S^1 are isomorphic to the direct sum of Q/Z with c copies of Q. Likewise $\mathbb{C}^*$ is isomorphic to the direct sum of S^1 with c or fewer copies of R^+, since both are isomorphic to

$$(Q/Z) \oplus \sum_a \oplus G_a.$$

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Portland State University

Also solved (or references given) by Stephen D. Bronn & Gilbert F. Orr, L. Richard Duffy, Lee Erlebach, Arnold D. Feldman, Steve Galovich, Jerrold W. Grossman & Suzanne Zeitman (partial solution), Chico Problem Group, William H. Gustafson, G. A. Heuer, P. B. Kronheimer (England), Thomas E. Moore, José Felipe Voloch (Brazil), and the proposers. There were three incorrect solutions.

Three solvers noted that the problem is solved in J. R. Clay, The punctured plane is isomorphic to the unit circle, J. Number Theory, vol. 1 (1969), 500–501. Heuer referred to the related MONTHLY problem 5013 (solution, vol. 70 (1963), 447).

The three incorrect solvers used the fact that $\mathbb{C}^*/S^1 \cong R^+ \not\cong 1$ to deduce $\mathbb{C}^* \not\cong S^1$.

A Stronger Trigonometric Inequality January 1982

1137. It is known that $\tan x + \sin x \geqslant 2x$ for $0 \leqslant x < \pi/2$, which is a stronger inequality than $\tan x \geqslant x$. Establish the still stronger inequality

$$a^2 \tan x (\cos x)^{1/3} + b^2 \sin x \geqslant 2xab$$

for $0 \leqslant x \leqslant \pi/2$. [*M. S. Klamkin, University of Alberta.*]

Solution: The stated inequality is valid for all pairs $(a, b) \in R \times R$ and all $x \in [0, \pi/2)$, with equality if and only if either $x = 0$ or $(a, b) = (0, 0)$.

Proof. The quadratic form

$$a^2(\tan x)(\cos x)^{1/3} - 2xab + b^2 \sin x$$

(in (a, b), for fixed x) has the discriminant

$$D = 4\left(x^2 - (\tan x)(\cos x)^{1/3}(\sin x)\right).$$

If $x = 0$, then $D = 0$ and the quadratic form vanishes. From now on we suppose $0 < x < \pi/2$. We shall prove that $D < 0$, which is equivalent to

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \qquad (1)$$

and which implies that the quadratic form is positive, except when $(a, b) = (0,0)$. Since

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} \text{ and } \frac{\sin x}{x} > 1 - \frac{x^2}{6} \text{ for } 0 < x < \frac{\pi}{2},$$

we need only prove that

$$1 - \frac{x^2}{2} + \frac{x^4}{24} < \left(1 - \frac{x^2}{6}\right)^3.$$

Since this is equivalent to $x^2 < 9$, hence true, the proof is complete.

It may be remarked that (1) is true for $0 < x \leqslant \pi$, since $\cos x \leqslant 0$ for $\pi/2 \leqslant x \leqslant \pi$.

ANDERS BAGER
Hjørring, Denmark

Also solved by the Bern-Kirchenfeld Literargymnasium Problem Solving Group (Switzerland), Victor E. Bloomfield, Erhard Braune (Austria), Zachary Franco (student), Chico Problem Group, Victor Hernandez (Spain), Hans Kappus (Switzerland), L. Kuipers (Switzerland), Beatriz Margolis (France), Vania D. Mascioni (Switzerland, student), David Paget (Australia), Lawrence A. Ringenberg, J. M. Stark, Michael Vowe (Switzerland), and the proposer.

Five solvers referred to D. Mitrinovic, Analytic Inequalities, Springer, 1970, p. 238, for a proof of (1). Braune used an extension of (1) (Mitrinović, p. 239) to derive the generalization

$$a^2(\tan x)^r(\cos x)^{r-1/3} + b^2(\sin x)^r \geqslant 2abx^s$$

for $0 < r \leqslant s \leqslant 1$ and $0 \leqslant x < \pi/2$ (with the appropriate limit from the left at $\pi/2$), and used this to show that

$$p_n + P_n > 4\pi R/(\cos(\pi/n))^{1/6},$$

where p_n and P_n are the perimeters of the inscribed and circumscribed regular n-gons of a circle of radius R. The Bern group proved the stronger inequality

$$a^2(\tan x)(\cos x)^{1/3} + b^2 \sin x \geqslant 2xab + \left(a(\tan x)^{1/2}(\cos x)^{1/6} - b(\sin x)^{1/2}\right)^2.$$

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q680. Obviously, x, y, and z cannot all be greater than 1; similarly, they cannot all be less than 1. If $x = 1$, then $y = z = 1$; similarly if $y = 1$ or $z = 1$. So we may assume without loss of generality that $x < 1 < z$ and $y \neq 1$. But if $y < 1$, the first equation is false; if $y > 1$, the second equation is false. (The conclusion also holds if one of a, b, c is 0 and the other two are positive.)

Q681. Let $f(n) = n + [2n/(n + 1983)]$ for $n \in N$. Alternatively, let $f(n) = n + \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(n - 1982\frac{1}{2})$. In each case, $f(n) = n$ or $f(n) = n + 1$ according as $n < 1983$ or $n \geqslant 1983$.

REVIEWS

Paul J. Campbell, Editor
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Kolata, Gina Bari, *Mathematician solves simplex problem*, Science 217 (2 July 1982) 39.

"For decades, investigators have wondered why the simplex algorithm [for linear programming] works so well in practice; now there is a proof that it has to." Stephen Smale (Berkeley) has shown that given a certain probability measure on the space of data, average-case behavior of the algorithm is far better than worst-case behavior.

Kolata, Gina, *Topologists startled by new results*, Science 217 (30 July 1982) 432-433.

Michael Freedman (San Diego) and Simon Donaldson (student, Oxford), have shown that there is more than one possible structure for "ordinary" four-dimensional space-time. Earlier, Freedman proved the four-dimensional Poincaré conjecture and showed that not all topological manifolds are differentiable. The new result is an existence proof by contradiction--so we don't still know what other possible space-times "look like."

Mandelbrot, Benoit B., The Fractal Geometry of Nature, Freeman, 1982; 460 pp.

Expansion and full fruition of the author's earlier Fractals: Form, Chance, and Dimension (1977). Hundreds of illustrations are here, including 10 in full color, all instances in a casebook to support Mandelbrot's manifesto: *There is a fractal face to the geometry of nature*. That the irregular and fragmented in nature, from coastlines to turbulence to galaxies, has yielded to mathematical description and analysis is yet another wonderful example of "the unreasonable effectiveness of the language of mathematics." Mandelbrot has banished technicalities to the end of the book and has let the expressive power of fractals emerge from the numerous computer-graphic illustrations. The result is a stunning accomplishment in natural philosophy.

Wieting, Thomas W., The Mathematical Theory of Chromatic Plane Ornaments, Dekker, 1982; vii + 369 pp.

Presents in detail the fundamental classification of plane ornaments into 17 types and systematically attacks the problem of classifying chromatic plane ornaments. For the latter, machine-generated counts are reported for 2 through 60 colors; and constructions are given for 2 through 8 colors, with a folio of illustrations for 4 colors (sadly, not printed in color). Readers need linear algebra, group theory, and the mature tolerance for symbolic argument developed in abstract algebra.

Tormey, Alan, and Tormey, Judith Farr, *Renaissance intarsia: the art of geometry*, Scientific American 247:1 (July 1982) 136-143, 154.

Intarsia is wood inlay. Unlike the inlay done in the Arab world, 15th century Italian intarsia represented three-dimensional objects in linear perspective. This article describes the method used by Leon Battista Alberti (1404-1472).

Williams, H. C., *The influence of computers in the development of number theory*, Computers and Mathematics with Applications 8 (1982) 75-93.

Discusses the ways computers have aided in the growth of number theory, specifically focusing on the topics of factoring, primality testing, Collatz's "Syracuse" problem, Abel's problem, Fermat's Last Theorem, the twin prime conjecture, the Riemann Hypothesis, and some problems from algebraic number theory. Up to the minute and excellent!

Guy, Richard K., Unsolved Problems in Number Theory, Springer-Verlag, 1981; xviii + 161 pp, $18.

First volume in Springer-Verlag's new series of Problem Books in Mathematics. This splendid volume has been "brewing" for 20 years and it includes extensive references at the end of each problem or article surveying a group of problems. Problems are classified roughly as involving prime numbers, divisibility, additive number theory, Diophantine equations, sequences of integers, or "none of the above." Sample: "Is there a triangle with integer sides, medians, and area? There are, in the literature, incorrect 'proofs' of impossibility, but the problem remains open."

Gelfond, A. O., Solving Equations in Integers, Imported Publications, 1981; 56 pp, $2 (P).

Concise and readable account of linear Diophantine equations, continued fractions, Pell's equation, approximation of algebraic numbers, infinite descent, and Fermat's Last Theorem for exponent 4. (Gelfond remarks, "High-school students will readily understand the subject-matter of the book." No doubt he is referring to Soviet high-school students, of whom 5 million will study calculus this year, almost 50 times as many as in the United States, and five times as many as college students in the U.S.)

Knuth, D. E., *Algorithms in modern mathematics and computer science*, in Ershov, A. E., and Knuth, D. E., Algorithms in Modern Mathematics and Computer Science, Springer-Verlag, 1981, pp. 82-99.

"What is the relation of algorithms to modern mathematics? Is there an essential difference between an algorithmic viewpoint and the traditional mathematical world-view? Do most mathematicians have an essentially different thinking process from that of most computer scientists?" Knuth considers nine examples of mathematics. He notes the absence there of two kinds of thinking: complexity considerations (cost of an algorithm) and the dynamic notion of state of a process (as expressed, for example, by the assignment operator), and muses: Are these what separate mathematicians from computer scientists?

Taylor, John R., An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements, University Science Books, 1982; xiv + 270 pp, $15, $9.50 (P).

Careful and easy-to-read analysis of handling of experimental errors, including their propagation in calculations (using calculus), statistical considerations, weighting by variance, least squares, and some probability distributions (normal, binomial, Poisson, chi-square). No mention of interval arithmetic or software available implementing it. Still, every science student should have it.

Smullyan, Raymond, The Lady or the Tiger? and Other Logic Puzzles, Including a Mathematical Novel that Features Gödel's Great Discovery, Knopf, 1982; ix + 226 pp, $13.95.

"Another scintillating collection of brilliant problems and paradoxes by the most entertaining logician and set theorist who ever lived. . . . You end up exploring that strange subterranean region, below mathematics, where Gödelian corridors lead in all directions to beautiful theorems about truth and provability."--Martin Gardner.

Bandelow, Christoph, Inside Rubik's and Beyond, Birkhäuser, 1982; vi + 125 pp, $3.95 (P).

With 20 color photos, plus attractive cartoons and graphics, and 30 pages on "The mathematical model" of the cube, this book is a cut above the dozen or so "memorize-my-algorithm" cube books. Variations and generalizations are briefly treated too, such as the supercube, Sam Loyd's famous 15-puzzle, Mèffert's Pyraminx, the 2x2x2 cube, the Magic Domino, and other potential magic regular polyhedra. Shouldn't every contemporary course in abstract algebra explore the cube?

Madachy, Joseph S., Ten Year Cumulative Index to the Journal of Recreational Mathematics, Baywood, 1982; 101 pp, $10 (P).

"The Subject and Title Index here is intended to be thorough"--in fact, it provides extraordinarily thorough coverage of this lively journal, dedicated to mathematics that does not require a college degree for its understanding. An author index is included too. (Note: The index is not sent automatically to subscribers; in fact, your library will need to be alerted to its existence.)

Prigogine, Ilya, From Being to Becoming: Time and Complexity in the Physical Sciences, Freeman, 1980; xix + 270 pp (P).

How has order emerged from chaos? What is the role of irreversible processes? The author, winner of the 1977 Nobel Prize for Chemistry, ponders these questions as he shows how elaborate structures arise at the macro and micro levels: patterns of circulation in the atmosphere, chemical waves in a reaction. The treatment uses a rich variety of mathematics (operators, bifurcations, attractors, probability theory) as it investigates dynamics, quantum mechanics, thermodynamics, self-organization, and equilibria.

Rucker, Rudy, Infinity and the Mind: The Science and Philosophy of the Infinite, Birkhäuser, 1982; x + 342 pp, $15.95.

Fascinating tour of the limits of science and the mind, in a mixture of mathematics and philosophy suitable for all lovers of paradoxes and ideas. Rucker has his own world-view to promote, which verges into mystical speculation; but a reader who doesn't go along with it can still enjoy liberation of the imagination and gain an insight into mathematics.

Temple, George, 100 Years of Mathematics: A Personal Viewpoint, Springer-Verlag, 1981; xii + 316 pp, $32.

The period in question is the immediately preceding century, from 1879 to the present. Temple has undertaken to offer a concept-based account of personally selected major developments in most directions of research, "with especial emphasis on their earlier stages." He concludes that "there are no longer any boundaries between the different 'branches' of the subject, which are now so closely interdependent that no mathematician can afford to specialize." Shouldn't a mathematical sciences graduate of today be familiar with, and have a perspective on, the major mathematical achievements of the last 100 years? Temple offers a readable and concise *vade mecum*.

NEWS & LETTERS

DISTINGUISHED SERVICE AWARD: EDWIN F. BECKENBACH

Edwin F. Beckenbach, who was well-known in the mathematical community for his research, teaching, writing, editing, and dedicated service to professional organizations, including the MAA, was named the recipient of the MAA Award for Distinguished Service and honored at the business meeting of the MAA in Denver this month. Known to this editor as a patient, encouraging, and skillful chairman of the MAA Committee on Publications (1971-82), he guided the MAA journals and series of books through this period of unprecedented growth and saw the newsletter *FOCUS* launched in 1981.

Ed was an enthusiastic participant at the summer MAA meetings in Toronto; for most of his friends, the close of the meetings was their last farewell. While vacationing after the meetings, he suffered a stroke and died on September 5, 1982.

Ivan Niven prepared the text of the citation that was read at the ceremony in Denver; this appears in the *Am. Math. Monthly*, February 1983. A few highlights follow.

Ed Beckenbach received his Ph.D. in 1931 from Rice University, a student of Lester R. Ford, Sr. He held positions in research and teaching at several institutions until he joined the mathematics faculty at UCLA, where he served for three decades. He played a central role in the establishment of the Institute for Numerical Analysis there, and was a founder and editor of the *Pacific Journal of Mathematics*. He wrote extensively on a wide range of mathematical topics; perhaps his books on inequalities, co-authored with Richard Bellman, are best known to MAA members.

Courtesy was a hallmark of Ed's demeanor, with one exception: he was merciless on the tennis court. Captain of the tennis team while at Rice, he played his favorite sport for six decades, most recently competing (with his wife, Alice) in national tournaments of "superseniors."

INEQUALITY REDISCOVERED

The underlying inequality in Hemenway's "Why your classes are larger than average" (this *Magazine*, May 1982, 162-164), which first appeared half a century ago, has also been expressed as "The average of velocity with respect to time is less than or equal to the average of velocity with respect to distance."

In [2] I presented a generalization of the inequality. Several responses [1] showed that over the years the inequality and its generalizations have appeared in such diverse areas as power series and distributive lattices.

As [1,2] show, it is intimately connected with the Schwartz inequality. I propose naming this best-known unknown inequality the "anonymous inequality" to give it the fame it deserves. It is available (though nameless) in [3], pp. 512-513 and in ex. 22, p. 114.

[1] R. A. Brualdi, Analysis, Amer. Math. Monthly, 84 (1977) 804.

[2] S. K. Stein, An inequality in two monotonic functions, Amer. Math. Monthly 83 (1976) 469-471.

[3] S. K. Stein, Calculus and Analytic Geometry, McGraw Hill, New York, 1982.

S. K. Stein
University of Calif., Davis

Hemenway has independently rediscovered a result which appears in articles by Scott Feld and myself:

Variation in class size, the class paradox, and some consequences for students. Research in Higher Education, v. 6, No. 3 (1977), 215,222.

Conflict of interest between faculty, students and administrators: Consequences of the class size paradox, in Frontiers of Economics, Gordon Tullock, ed., v. 3 (1980).

Bernard Grofman
University of Calif., Irvine

1983 OHIO SHORT COURSE

Carl Pomerance and Samuel S. Wagstaff, Jr. will be the principal lecturers for the 1983 MAA Ohio Section short course to be held at Kent State University, June 16-18, 1983. The course will give an introduction to modern factoring and primality testing algorithms, culminating with the new Adleman-Rundy-Cohen-Lenstra tests.

The lectures will assume a general understanding of undergraduate modern algebra, and the workshops will assume the basic fundamentals of Fortran or WATFIV programming. Registration fee is $30. For further information, contact:

Jacqueline Parsons
Conference Bureau
211A Kent Student Center
Kent State University
Kent, OH 44242

Tel: (216) 672-3161

VisUMAP PROJECT

VisUMAP (the Visual Mathematics and its Applications Project) is a project of the Consortium for Mathematics and its Applications (COMAP) which proposes to produce 30 half-hour television programs, plus text and test materials, that demonstrate contemporary, real-world applications of mathematics.

The programs, designed for use as an Introduction to Mathematics or Mathematics for Liberal Arts offering, will deal with mathematics applications in six primary areas: Management Science, Politics, Growth, Computers, Choice and Chance, and the Physical Universe. Each topic area will be introduced by an overview program, and then explored in further detail by three to four classroom-styled programs.

The video production, in addition to traditional approaches--dramatization, narration, location shooting, etc.--will include the use of a variety of video techniques, such as cel animation and computer-generated graphics.

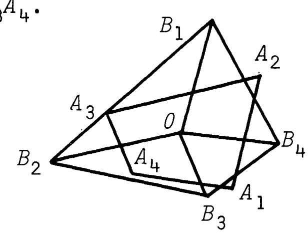
For further information, contact: Solomon A. Garfunkel, Project Director, or Donald S. Berman, Executive Producer, VisUMAP/COMAP, 55 Chapel St., Newton, MA 02160.

SOLUTIONS TO 1982 CANADIAN AND INTERNATIONAL OLYMPIADS

We appreciate the work of Loren Larson, St. Olaf College, who prepared the solutions to the 1982 Canadian Olympiad. His solutions to the 1982 IMO are based on those prepared by the MAA Committee on High School Contests, which are available (for 50¢) in a pamphlet from Dr. Walter E. Mientka, 917 Oldfather Hall, U. of Nebraska, Lincoln, NE 68588-0322.

1982 CANADIAN MATH OLYMPIAD

1. In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1,2,3$ and 4 ($A_5 = A_1$). Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.



Sol. Set $B_5 = B_1$, $A_5 = A_1$, and $A_6 = A_2$. From the geometry of the problem, Area B_iOB_{i+1} = Area $A_iA_{i+1}A_{i+2}$ for $i = 1,2,3,4$. Therefore,

$$\text{Area } B_1B_2B_3B_4 = \sum_{i=1}^{4} \text{Area } B_iOB_{i+1} = \sum_{i=1}^{4} \text{Area } A_iA_{i+1}A_{i+2} = 2 \text{ Area } A_1A_2A_3A_4$$

2. If a, b and c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,

(i) show that a, b and c are distinct;
(ii) show that

$$\frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$

is an integer.

Sol. (i) We have

$$a + b + c = 1, \quad (1)$$
$$ab + ac + bc = -1, \quad (2)$$
$$abc = 1. \quad (3)$$

Suppose two of the roots are equal, say $a = b$. Substitute $c = 1 - 2a$ from (1) into (2) to get $a = 1$ or $a = -1/3$. Then, from (1) we find $c = -1$ or $c = 5/3$ respectively. But these values for a, b, c do not satisfy (3), a contradiction.

(*ii*) For each nonnegative integer k, let $S_k = a^k + b^k + c^k$. We have $S_0 = 3$, $S_1 = 1$, $S_2 = S_1^2 - 2(ab + ac + bc) = 3$, and for $k > 2$,

$$S_k = S_{k-1} + S_{k-2} + S_{k-3}$$

(substitute a, b, c into $x^k = x^{k-1} + x^{k-2} + x^{k-3}$ and add). It follows by induction that S_k is an integer for each k.

Let $n = 1981$, and let $S(n+1)$ denote the sum in question. The proof then is a consequence of the following identity.

$$S(n+1) = \sum_{i=0}^{n} (a^{n-i} b^i + b^{n-i} c^i + c^{n-i} a^i)$$

$$= 2S_n + \sum_{i=1}^{n-1} (a^{n-i}b^i + b^{n-i}c^i + c^{n-i}a^i)$$

$$= 2S_n + \sum_{i=1}^{\frac{n-1}{2}} (a^{n-i}b^i + a^i b^{n-i} + b^{n-i}c^i + b^i c^{n-i} + c^{n-i}a^i + c^i a^{n-i})$$

$$= 2S_n + \sum_{i=1}^{\frac{n-1}{2}} (S_{n-i}S_i - S_n)$$

$$= (2 - \frac{n-1}{2})S_n + \sum_{i=1}^{\frac{n-1}{2}} S_{n-i}S_i .$$

3. Let R^n be n-dimensional Euclidean space. Determine the smallest number $g(n)$ of points of a set in R^n such that every point in R^n is at an irrational distance from at least one point in that set.

Sol. Clearly, $g(1) = 2$. Consider $n > 1$. To show $g\ n) > 2$ we must show that for any two points A and B in R^n there is a third point P whose distance from A and B are each rational. There is no loss in generality in supposing that $A = (0,0,..,0)$ and $B = (a,0,..,0)$ for some $a > 0$. Let k be any rational greater than $a/2$. Then it is easy to check that the point

$P = (a/2, \sqrt{(4k^2 - a^2)/2}\ )$ is distance k away from each of A and B.

Now let $A = (0,0,\ldots,0)$, $B = (\sqrt{2},0,\ldots,0)$, $C = (\sqrt{3},0,\ldots,0)$. We will show that every point in R^n is an irrational distance away from either A, B, or C.

Suppose that $P = (x_1, x_2, \ldots, x_n)$ is a rational distance away from each of A, B, and C. Then there are rationals r, s, t such that

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r,$$

$$(x_1 - \sqrt{2})^2 + x_2^2 + \ldots + x_n^2 = s,$$

$$(x_1 - \sqrt{3})^2 + x_2^2 + \ldots + x_n^2 = t.$$

It follows that

$$-2\sqrt{2}\, x_1 + 2 = s - r,$$

$$-2\sqrt{3}\, x_1 + 3 = t - r,$$

or equivalently,

$$-2x_1 = \frac{2 + r - s}{\sqrt{2}},$$

$$-2x_1 = \frac{3 + r - t}{\sqrt{3}}.$$

This implies that

$$\sqrt{3}\,(2 + r - s) = \sqrt{2}\,(3 + r - t),$$

an impossibility. It follows that $g(n) = 3$ for all $n \geq 2$.

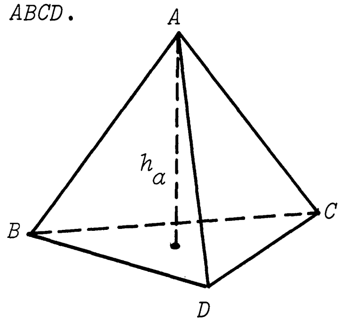
4. Let p be a permutation of the set $S_n = \{1,2,\ldots,n\}$. An element $j \in S_n$ is called a fixed point of p if $p(j) = j$. Let f_n be the number of permutations of S_n having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

Sol. From the recurrences $f_n = (n-1)\ (f_{n-1} + f_{n-2})$ and $g_n = nf_{n-1}$, we get

$$f_n - g_n = g_{n-1} - f_{n-1}.$$

The result now follows by induction.

5. The altitudes of a tetrahedron $ABCD$ are extended externally to points A', B', C' and D' respectively, where $AA' = k/h_a$, $BB' = k/h_b$, $CC' = k/h_c$ and $DD' = k/h_d$. Here k is a constant and h_a denotes the length of the altitude of $ABCD$ from vertex A, etc. Prove that the centroid of the tetrahedron $A'B'C'D'$ coincides with the centroid of $ABCD$.



Sol. Let K denote the volume of the tetrahedron. Then $h_a \cdot$ Area $\Delta BCD = K/6$, and $|\vec{BC} \times \vec{BD}| = \frac{1}{2}$ Area $\Delta BCD = \dfrac{K}{12h_a}$.

Given $|AA'| = k/h_a$, it follows that

$$\vec{BC} \times \vec{BD} = \left(\frac{K}{12h_a}\right)\left(\frac{\vec{AA'}}{|\vec{AA'}|}\right) = \left(\frac{K}{12h_a}\right)\left(\frac{h_a}{k}\right)\vec{AA'},$$

or equivalently, $\vec{AA'} = \dfrac{12k}{K}(\vec{BC} \times \vec{BD})$.

In a similar manner, $\vec{BB'} = \dfrac{12k}{K}(\vec{AD} \times \vec{AC})$

$\vec{CC'} = \dfrac{12k}{K}(\vec{BD} \times \vec{BA})$, $\vec{DD'} = \dfrac{12k}{K}(\vec{BA} \times \vec{BC})$.

Using these, $\vec{AA'} + \vec{BB'} + \vec{CC'} + \vec{DD'}$

$$= \frac{12k}{K}(\vec{BC} \times \vec{BD} + \vec{AD} \times \vec{AC} + \vec{BD} \times \vec{BA} + \vec{BA} \times \vec{BC}) = \frac{12k}{K}\Big[\vec{BC} \times \vec{BD} + (\vec{BD} - \vec{BA}) \times (\vec{BC} - \vec{BA}) + \vec{BD} \times \vec{BA} + \vec{BA} \times \vec{BC}\Big] = \ldots = 0.$$

Let O be a fixed point, let $\vec{A} = \vec{OA}$, etc. Then Centroid $A'B'C'D' = (\vec{A}' + \vec{B}' + \vec{C}' + \vec{D}')/4 = \Big[(\vec{A} + \vec{AA'}) + (\vec{B} + \vec{BB'}) + (\vec{C} + \vec{CC'}) + (\vec{D} + \vec{DD'})\Big]/4 = (\vec{A} + \vec{B} + \vec{C} + \vec{D})/4 + (\vec{AA'} + \vec{BB'} + \vec{CC'} + \vec{DD'})/4 = (\vec{A} + \vec{B} + \vec{C} + \vec{D})/4 =$ Centroid $ABCD$.

XXIII INTERNATIONAL MATHEMATICAL OLYMPIAD
JULY 9-10, 1982
BUDAPEST, HUNGARY

1. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m,n,

$f(m + n) - f(m) - f(n) = 0$ or 1;

$f(2) = 0$, $f(3) > 0$,

and $f(9999) = 3333$.

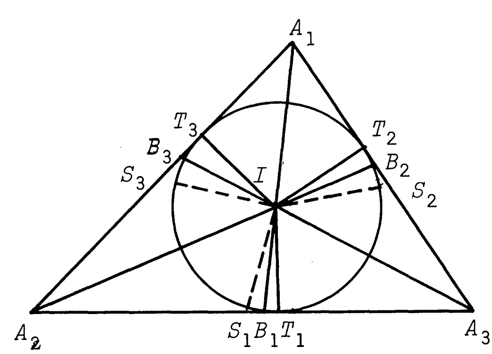
Determine $f(1982)$.

Sol. It is easy to show that $f(1) = 0$ and $f(3) = 1$. An easy induction shows that $f(3n) \geq n$, and furthermore, if $f(3n) > n$ for any n, then $f(3m) > m$ for all $m > n$. Since $f(9999) = 3333$, we must have $f(3n) = n$ for all $n \leq 3333$.

Thus we have $1982 = f(3 \cdot 1982) \geq f(2 \cdot 1982) + f(1982) \geq 3f(1982)$. It follows that $f(1982) \leq \dfrac{1982}{3} < 661$. Also, $f(1982) \geq f(1980) + f(2) = 660$. Therefore, $f(1982) = 660$.

2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1,a_2,a_3 (a_i is the side opposite A_i). For all $i = 1,2,3$, M_i is the midpoint of side a_i, T_i is the point where the incircle touches side a_i, and the reflection of T_i in the interior bisector of A_i yields the point S_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

Sol. Consider the following diagram, where A_iB_i are the angle bisectors as shown.



We have,

$\angle T_1IT_3 = 180° - A_2$,

$\angle T_3B_3A_3 = A_2 + A_3/2$,

$\angle T_3IB_3 = 90° - (A_2 + A_3/2)$,

$\angle T_3IS_3 = 180° - (2A_2 + A_3)$,

$\angle S_3IT_1 = \angle T_3IT_1 - \angle T_3IS_3 = A_2 + A_3$.

In a similar way, $\angle S_2IT_1 = A_2 + A_3$. Thus, $\angle S_3IT_1 = \angle S_2IT_1$, so that chord S_1S_2 is parallel to side A_2A_3. Similarly, $S_2S_3 || A_2A_3$ and $S_3S_1 || A_3A_1$.

We know that $M_1M_2 || A_1A_2$, etc. and it follows that triangles $M_1M_2M_3$ and $S_1S_2S_3$ have parallel sides. Thus, they are related by a translation if they are congruent or by a homothety if they are not. Now, $S_1S_2S_3$ is inscribed in the incircle, whereas $M_1M_2M_3$ is inscribed in the nine-point circle. Since $A_1A_2A_3$ is scalene, triangles $S_1S_2S_3$ and $M_1M_2M_3$ are homothetic, and M_1S_1, M_2S_2, and M_3S_3 are concurrent.

3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1 \text{ and for all } i \geq 0,$$
$$x_{i+1} \leq x_i .$$

(a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4 \text{ for all } n.$$

Sol. (a) Suppose we have a positive number c_n such that for every sequence $x_0 \geq x_1 \geq \cdots \geq x_n \cdots > 0$ the following inequality holds:

$$\frac{x_0^2}{x_1} + \cdots + \frac{x_{n-1}^2}{x_n} \geq c_n x_0 .$$

Notice that we may choose $c_1 = 1$. Also, for every such sequence,

$$\frac{x_0^2}{x_1} + \left(\frac{x_1^2}{x_2} + \cdots + \frac{x_n^2}{x_{n+1}}\right) \geq$$

$$\frac{x_0^2}{x_1} + c_n x_1 \geq 2\sqrt{\frac{x_0^2}{x_1} \cdot c_n x_1} = 2x_0\sqrt{c_n} .$$

Therefore, we may choose $c_{n+1} = 2\sqrt{c_n}$. This recurrence relation implies that

$$c_n = 2^{1 + 1/2 + \cdots + 1/2^{n-2}}$$
$$= 4 \cdot 2^{-1/2^{n-2}} .$$

For c_n to exceed 3.999, we must find n so that $\left(\frac{4.000}{3.999}\right)^{2^{n-2}} > 2$.

We have $\left(\frac{4.000}{3.999}\right)^{2^{n-2}} > \left(1 + \frac{1}{4000}\right)^{2^{n-2}}$

$> 1 + \frac{2^{n-2}}{4000}$, which is > 2 when $n = 14$.

(b) Let $x_n = 2^{-n}$.

4. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x,y), then it has at least three such solutions.

Show that the equation has no solution in integers when $n = 2891$.

Sol. (a) It is easy to check that if (x,y) is a solution, then so are $(y-x,-x)$ and $(-y, x-y)$, and these are distinct.

(b) Suppose that x and y are integers such that $x^3 - 3xy^2 + y^3 = 2891$. Then $x^3 - 3xy^2 + y^3 \equiv 2 \bmod 9$. We shall construct a contradiction to this congruence.

We have $x^3 - 3xy^2 + y^3 \equiv x^3 + y^3 \equiv 2 \bmod 3$. There are three cases:

(i) If $x \equiv 0 \bmod 3$ then $y \equiv 2 \bmod 3$;
(ii) If $x \equiv 1 \bmod 3$ then $y \equiv 1 \bmod 3$;
(iii) If $x \equiv 2 \bmod 3$ then $y \equiv 0 \bmod 3$.

In case (i), $x = 3s$ and $y = 3t - 1$, and on substitution, we have $x^3 - 3xy^2 + y^3 = 27s^3 - 9sy^2 + 27t^3 - 27t^2 + 9t - 1 \equiv -1 \bmod 9$, a contradiction. For case (iii), we have a similar result and contradiction.

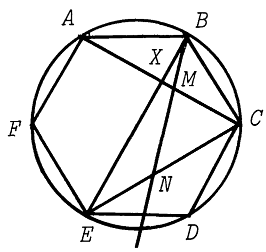
In case (ii), we know from part (a) that $(y-x, -x)$ is also a solution, but this solution has the form of case (i), which we have shown is impossible.

5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r .$$

Determine r if B, M, and N are collinear.

Sol. Assume that each side of the hexagon has length one, and that B, M, and N are collinear. Let X denote the intersection of AC and BE.



By Menelaus' Theorem, applied to triangle CEX, we know that

$$\frac{CN}{NE} \cdot \frac{EB}{BX} \cdot \frac{XM}{MC} = -1.$$

Let $\frac{CN}{CE} = r$. Our aim is to express each of the distances in Menelaus' formula in terms of r. We have, $CE = \sqrt{3}$, $CN = \sqrt{3}r$, $NE = CE - CN = \sqrt{3}(1 - r)$, $EB = 2$, $BX = -1/2$, $AM = r\sqrt{3}$, $MC = AC - AM = \sqrt{3}(1 - r)$, $XM = \sqrt{3}/2 - MC = \sqrt{3}/2 - \sqrt{3} + r\sqrt{3}$ $= \sqrt{3}(r - 1/2)$.

Substituting into Menelaus' formula, we have

$$\frac{\sqrt{3}r}{\sqrt{3}(1 - r)} \cdot \frac{2}{- 1/2} \cdot \frac{\sqrt{3}(r - 1/2)}{\sqrt{3}(1 - r)} = -1,$$

and this yields $r = 1\sqrt{3}$.

6. Let S be a square with sides of length 100 and let L be a path within S which does not meet itself and which is composed of linear segments A_0A_1, A_1A_2, ... ,$A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than 1/2. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1 and the length of that part of L which lies between X and Y is not smaller than 198.

Sol. Let the distance between a point P on the side of the square and a point Q on the polygonal path be $d(PQ)$, and the length of the polygonal path from A to B be $s(AB)$. If $s(A_0A) < s(A_0B)$, we shall write $A < B$.

Let S_1, S_2, S_3, S_4 be the vertices of the square. On L take points S_1', S_2', S_3', S_4' such that $d(S_iS_i') \leq 1/2$. We can assume that $S_1' < S_4' < S_2'$.

Now let L_1 be the set of all X on L such that $X \leq S_4'$, and let L_2 be the set of all X on L such that $X \geq S_4'$. Consider side S_1S_2. There is a subset of L_1' of S_1S_2 whose points are distant $\leq 1/2$ from L_1, and a subset L_2' of S_1S_2 whose points are distant $\leq 1/2$ from L_2. (Since L_1' includes S_1 and L_2' includes S_2, neither set is empty.)

The union of L_1' and L_2' is the side S_1S_2, and, because of the condition on the distances, the intersection of L_1' and L_2' is not empty. Let M be a point common to L_1' and L_2'.

Now select a point X in L_1 and a point Y in L_2 such that $d(MX) \leq 1/2$ and $d(MY) \leq 1/2$. Then $d(XY) \leq 1$. Also, $X < S_4' < Y$, and

$s(XY) = s(XS_4') + s(S_4'Y) \geq 99 + 99 = 198.$

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